How to Model?

Tony Hürlimann

info@matmod.ch

November 12, 2023

Abstract

This paper has three parts. It gives first a short introduction on the modeling process, explains very briefly its stages in order to learn how to model. In a second part a number of small problems with solutions in a third part are exposed by which the reader can exercise the skills of modeling. To be useful, the reader should really try hard to solve these problems and look up the solution only after trying. It is the only way to acquire the skill of modeling. Three problems are from [4] which is an excellent book with more than 150 puzzles. Many Internet pages and books are available with puzzles, one of the oldest is [3]. Martin Gardner has also published a number of mathematical puzzles in book form. The focus here is mainly on mathematical models, hence the third part of the problems have also mathematical formulations if appropriate. Many modeling examples of puzzles can also be found in my puzzle and other book: See My Modeling Books.

1 Introduction

Modeling is an art! But modeling is also a skill that can be learnt. Every problem to be solved is new, therefore, there is no unique recipe to model and solve it, but there are guidelines. The most important ability, however, is *to practise*, *to practise* and *to practise*. Looking at the problems, try to solve them, analyzing and reproducing how other persons solved them. In this way, the learner gains experience and security. Also you should learn the theoretical concepts going along the modeling process. For example, if the concept of "graph" is used, you must learn the basic concepts of graph theory, if modeling a "linear mathematical model", the basic concepts of linear algebra must be learnt.

Basically, The process of modeling goes in stages:

Recognition – formulation – solution – validation

Recognition is the stage where the problem is identified and the importance of the issue is perceived; *formulation* is the most difficult step: the model is set up and a mathematical structure is established; *solution* is the stage where an algorithm is chosen or conceived to find a solution; finally, the model must be *interpreted* and *validated* in the light of the original problem: Is it consistent? Does it correspond to reality? etc. Modeling, however, is essentially an iterative process where the different stages must be repeated. The reasons are manifold: A first attempt often does not produce a satisfactory model to the original problem. It is often too crude or simply inaccurate or even wrong in representing the original problem. When inaccurate, the model has to be *modified*; when too crude, it must be *refined*; when too inappropriate the model has to be *rejected*. This modify-refine-reject process is very important and it is the rule and not the exception. Keep this in mind! Every step is new and a creative process that can go in a wrong direction, so *perseverance* is another prerequisite a modeler must acquire to be successful. Going from stage to stage also means discovering – uncovering – new structures and connections that were previously unknown to the model builder. This is unavoidable as the modeling process is not a mechanical activity, where the intermediate and final goals are known in advance. The process is fundamentally non-predictable. The evolution of a model is not unlike that of paradigms described by the history of science [8]. "Normal" periods of successive small improvements and corrections are followed by "periods of crisis", during which the basic statements are radically questioned and eventually replaced. For these reasons, models should always be regarded as tentative and subject to continued evaluation and validation. Remember it is easy to falsify, but not easy to verify a model! Let us have a closer look at the four stages.

1.1 Recognition and Specification of the Problem

First of all, it must be said that very little progress can be made in modeling without fully understanding the problem that has to be modeled. If the modeler is not familiar with the real problem, she will never be able to build a model of it. Traditionally, not much mathematics is involved at this stage. The problem is studied for an extended time, data is collected and empirical laws or guidelines are formulated before a systematic effort is made to provide a supporting model. The first mistake comes from an erroneous idea that the final and formalized model and its solution are more important than this "preliminary" stage, although a careful and accurate study of the problem is essential. The second mistake often arises because many modelers think that using *some* formalism is more important than using an appropriate notation that reflects the problem. Yet the most sophisticated formalism is useless if it does not mirror some significant features of the underlying problem. Often, the modelers only have access to a limited arsenal of modeling tools, and they apply what they have at hand. Appropriateness comes second, so the erroneous approach often goes!

Hence, the very beginning of the modeling process needs expertise and a good eye to filter out unimportant factors. As difficult, and sometimes as tedious as the modeling process may be, there is no way around it: It is impossible to specify a problem without *setting up an informal model*. Any specification is a model using some language or symbolism. The first formulation is mostly in natural language. Sketches or drawings usually accompany the formulation. At this stage, the objectives should be clearly stated. A plan on how to attack

the problem has to be put forward. In textbooks on problem solving in applied mathematics, this first stage is assumed, but in facing a real problem, it is often not even clear what the problem consists of. Unless the modeler gets this right at the start, it is of little use to build a formalized model.
Summary: Repeat the problem in your own words! Make sketches. Are there data? What is the goal? What are the conditions? Explain the problem to another person! Do not stop before you understand the problem entirely. Order the facts! Name the components! Do not hesitate to restart!

1.2 Formulation of the (Mathematical) Model

Mathematics is the science of searching for patterns and for structures. This is intellectually the most rewarding activity, it is undoubtedly creative. Translation a problem into mathematical notation uses mathematical knowledge and skills as well as problem expertise. But it also requires something more: the talent to recognize what is essential in a particular context and to discard the irrelevant, as well as an intuition as to which mathematical structure is best suited to the problem. This intuition cannot be learned as we learn how to solve mathematical equations. Like swimming, this skill can be acquired by observing others and then trying by oneself. Hence, it is difficult to provide an adequate formal description on the model formulation process.
Since the focus here is on *mathematical models* as defined in another paper (see [7]). On an abstract level, there is no doubt about what has to be done: defining the variables x and finding the constraints $R(x)$ for the model, as well as collecting the data and the parameters p. On a more concrete level, one can only give some guidelines, such as:

- Establish a clear statement of the objectives
- Recognize the key variables, group them (aggregate)
- Identify initial states and resources
- Draw flow diagrams
- Do not write long lists of features
- Simplify at each step
- Get started with mathematics as soon as possible
- Know when to stop.

Pólya, in his classical work [1] (see also [2]) about problem solving, tried to give heuristic strategies to attack mathematical problems – an essay that every person who does serious modeling should read. Pólya has summarized them as follows:

1. Understanding the problem: What is (are) the unknown(s)? What are the data? What are the conditions or constraints? Are they contradictory or are some of them redundant?

2. Devising a plan of attack: Do you know of a related problem? Can you restate the problem? Can you solve a part, a special case or a more general problem? Did you use all the data and all the essential notions of the problem?

3. Carrying out the plan: Solve the related problems. Check each step.

4. Looking back: Do you really believe the answer?

In the second part of this paper several examples are given.

1.3 Solving the Mathematical Model

Once the model formulation step has been completed, the mathematical model has to be solved, i.e. the model must be transformed using calculation in such a way as to eliminate the unknowns. We say that a model is solved if a numerical or symbolical expression containing only parameters and numbers is assigned to each variable so that all constraints are fulfilled.
The solution step can be very straightforward, as in most examples in the third part of this paper. Most of the time this is not the case. It is easy to formulate innocent looking models that are very difficult to solve. For certain model classes a well understood mathematical framework and numerical software exist to solve them. If not, the model must be reformulated to be tractable. Often it is not necessary to reformulate an intractable model; one can use *simulation techniques* or *heuristics* to find approximate – hopefully good – solutions to the model. Anyway, in all but trivial cases the solution cannot be carried out by hand, and one must develop computer programs to execute the lengthy calculations. But even if the mathematical theory of a problem is straightforward, the invention of an appropriate algorithm can require considerable ingenuity. Furthermore, it is not easy to write numerically stable and reliable software. Therefore, for many problem classes available software packages – called *solvers* – have been built by expert numerical analysts.
When using a solver, the model must be put into a form that can be read by the appropriate solver. This is a laborious and error pruned task especially for large models with lot of data involved. Data must be read and filtered correctly from databases, the model structure must be reformulated, and the generated output must be a correct form for input into the solver. For this task, modeling tools such as (mathematical) *modeling languages* exist that can simplify greatly this task. (I have more to say about modeling languages in a separate paper: [6].)

1.4 Validation and Interpretation

Creating and building a model is a creative process where "everything goes", but checking and validating the model is a completely different affair. Validation, in fact, is a continuous process and takes place at every stage not only in the final part. By validation we mean the process of checking the model against reality, and the strongest types of validation is *prediction.*[1] But validation is much more than inspecting the similarities or dissimilarities of the model against reality. It comprises a variety of investigations that help to make the model more or less *credible*. This consists of many questions that should be answered, such as: Is the model mathematically consistent? Is the data used by the model consistent? Is it commensurable? Does the model work as intended? Does it "predict" known solutions using historical data? Do experts in the field come to the same conclusions as the model? How does the model behave in extreme or in simplified cases? Is the model numerically stable? How sensitive is the model to small changes in input data? Let's go through several elements:

1. **Logical Consistence**: Of course, a model must be free of contradiction, since from a inconsistent statement everything follows ("ex falso quodlibet"). This is trivial! However, in a practical model building process, inconsistencies arise in a natural way, mostly because of conflicting goals or hard constraints[2].

2. **Data type consistency**: This issue has been discussed in detail and over many years by the programming language community. Data type checking means to check the domain membership of a value before it is assigned to a storage object in computers.

3. **Unit Type Consistency**: Another issue is to check whether the data in expressions are commensurable. In a modeling context, dimensional checking is of utmost importance, since most quantities are measured in units (dollar, hour, meter, etc.). Experiences in teaching operations research reveal that one of the most frequent errors made by students when modeling is inconsistency in the units of measurement. In physics and other scientific applications, as well as technical and commercial ones, using units of measure in calculating has a long tradition. It increases both the reliability and the readability of calculations.

4. **User defined data checking**: Do I have the right data? Are the data correct and consistent? Are the data complete? Often data checking is a huge task itself. But it is necessary, otherwise we get a garbage-in garbage-out model, that is, the model is only useful if the data make sense. The data should also be separated as much as possible from the

[1]The discovery of the planet Neptune is an often quoted example. Since the discovery of the planet Uranus in 1781, perturbations had been noted in its orbit. An explanation within Newton's gravitational model which could fit these observations was the presence of a previously unobserved planet. A telescope at Berlin was then pointed in the right direction at the right time, on the basis of Le Verrier's extended calculations, and a new planet was found: Neptune. This example shows the predictive power of a good model. But history also shows how careful we should be about making general conclusions: Later, when irregularities in Mercury's orbit were discovered, a similar explanation with a new planet, Vulcan, was proposed; but Vulcan was never discovered! These irregularities were only explained by Einstein's general theory of relativity.

[2]When we say that a budget should by $1'000'000$, then we normally do not mean that it must be *exactly* that amount, we mean "about 1'000'000". Many model examples show how to handle these kind of situations (one is in my puzzle book, see library)

model structure (see [7]). In this way, a data set can easily be replaced by another data set. Applying various data set (smaller data set or random sets, etc., in a prototyping preliminary stage) can largely help to find inconsistencies.

5. **Simplicity Considerations**: Can the model be simplified or reduced? The principle of Ockham's Razor also applies to modeling. To paraphrase Einstein: The model should be as simple as possible, but not simpler.

6. **Solvability Checking**: Formulate model that are difficult to solve is easy. One can try various solvers, but sometimes one can modify the model – for instance by adding trivial constraints – to help the solver to find a solution in shorter time. These techniques need a lot of experiences and cannot be exposed here (without going into the details: an small and simple example is the Pigeon hole model pihole).

The model creation process is a difficult undertaking. I barely touched a few highlights. What is important is to practice to gain experience and to study in parallel the branches in mathematics used in a particular context. What follows in a second part of this paper is a number of rather easy problems, that the reader can try to solve. First the problems are formulated, after a possible solution is presented with some remarks on modeling exposed in the previous sections.

1.5 The Modeling Life-Cycle

The model building process can also be viewed as cycle as depicted schematically in Figure 1.

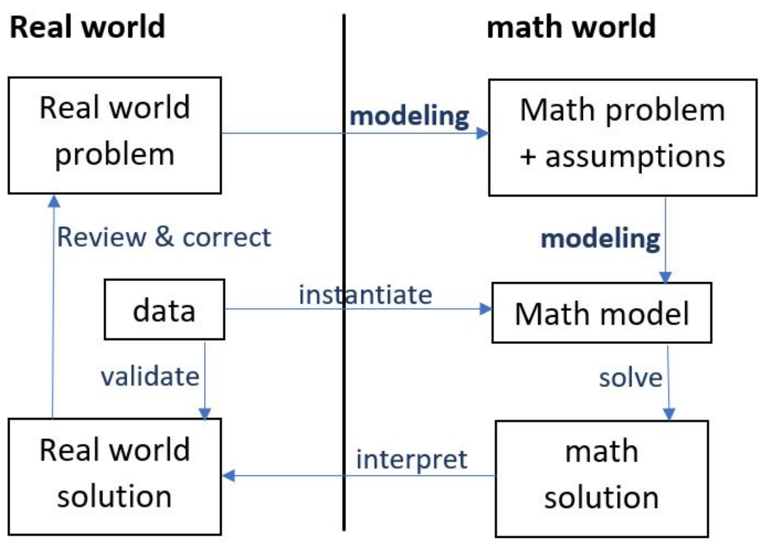


Figure 1: The Modeling-Life-Cycle

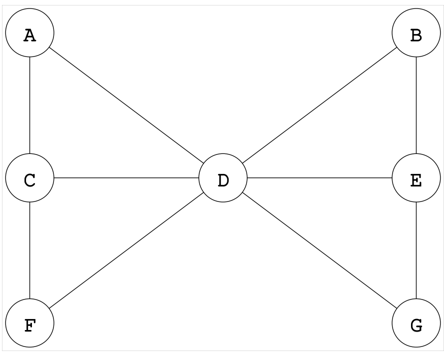
The starting point is a "real world problem" (where "real world" is meant to be very general – it could be an abstract problem, like the 4-color problem of a

planar graph, for example). The first step in modeling is to translate this problem into a precisely defined mathematical problem. Maybe a similar problem exists already and the modeler only need to "match" the real problem to the mathematical symbolism. Sometimes, however, a new theory or formalism must be invented in order to capture the problem – Newton invented the infinitesimal calculus to explain the motion and dynamic change, such as the orbits of planets, the motion of fluids, etc.
To solve a concrete "real" problem, the mathematical problem must be enriched with *assumptions* and *data*. For example, to find the shortest path from my home to the office, one needs to known what kind of vehicle to use, what are the distances, is there a traffic jam, etc. Hence, a second modeling step is to collect all the needed data, to add implicitly or explicitly all the relevant assumptions to build a concrete mathematical model, that is, to instantiate the model, which then can be solved. To solve a problem mean to transform the problem using mathematical operations and rules until the unknowns become known values. This solving step is mostly done by an algorithm – called *solver* – and executed on a computer.
Once the solution is known, it must be interpreted in the light of our "real" problem and validated by the data. Does the result make sense? Does it match the data? If not, we must review and correct our steps. Maybe one needs to modify the data, the assumptions, or change the whole approach all together. Sometimes completely new insights in the problem are discovered which can lead to a complete new "real problem" which appears to be more promising. An the whole modeling life cycle begins afresh.

2 The Problems

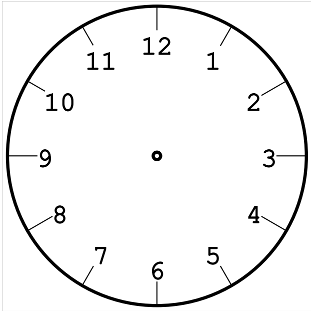
2.1 Problem 1: Seven Digits Puzzle

Use each digit from 1 to 7 exactly once, and place them into the circles of the Figure at the right side in such a way that the sum along each of the five straight lines is the same (for example, A-C-F is a straight line).



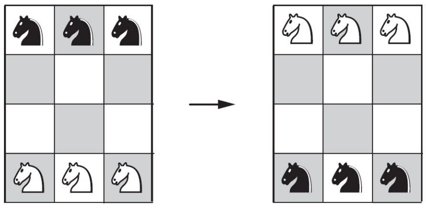
2.2 Problem 2: The Clock Puzzle

Divide the clock with a straight cut into two parts such that the sum of the numbers in both parts are equal?



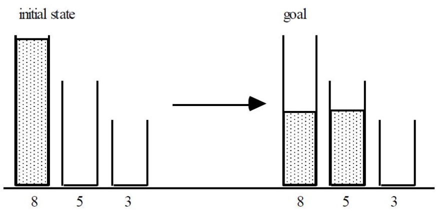
2.3 Problem 3: The 6-Knights Puzzle

Move the knights from a initial position (left part of the Figure) using the chess rules in a 3×4 board to a final position (right part). The puzzle is from [4].



2.4 Problem 4: The 3-jug Problem

There are three jugs with capacities of 8, 5, and 3 liters. Initially the 8-liter jug is full of water, whereas the others are empty. Find a sequence for pouring the water from one jug to another such that the end result is to have 4 liters in the 8-liter jug and the other 4 liters in the 5-liter jug. When pouring the water from a jug A into another jug B, either jug A must be emptied or B must be filled.



2.5 Problem 5: The Fake Coin Puzzle I

There are 8 identical-looking coins; one of these coins is counterfeit and is known to be lighter than the genuine coins. What is the minimum number of weighings needed to identify the fake coin with a two-pan balance scale without weights? The puzzle is from [4].

2.6 Problem 6: The Fake Coin Puzzle II

There are 12 coins identical in appearance; either all are genuine or exactly one of them is fake. It is unknown whether the fake coin is lighter or heavier than the genuine one. You have a two-pan balance scale without weights. The problem is to find whether all the coins are genuine and, if not, to find the fake coin and establish whether it is lighter or heavier than the genuine ones. Design an algorithm to solve the problem in the minimum number of weighings. The puzzle is from [4].

2.7 Problem 7: The Horse Race Puzzle

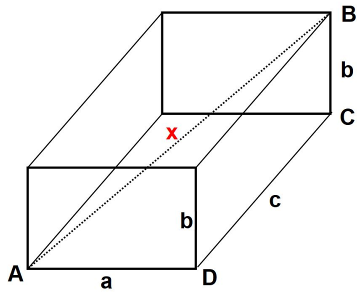
There are 25 horses. What is the minimum number of races needed to identify the 3 fastest horses? Up to 5 horses can race at a time, no watch is allowed.

2.8 Problem 8: Adding Numbers

Find the sum of all integers from 1 to 1000. Then find a formula for the sum of all integer from 1 up to an number $n > 1$ and proof it by induction.

2.9 Problem 9: 3d-Diagonal

Find the diagonal x of a rectangular parallelepiped of which the length c, the width a, and the height b are known.



2.10 Problem 10: Square Inside a Triangle

Inscribe a square in a given triangle. Two vertices of the square are on the base line of the triangle, and the other two vertices touch the two sides of the triangle. Are they more then one possibilities?

2.11 Problem 11: Birthday Paradox

Given a party of n persons, what is the probability $P(n)$ that at least 2 persons have the same birthday? For instance, if 30 persons are invited to a party, is it worthwhile to make a bet that two persons have the same birthday?

2.12 Problem 12: Length of Loch Ness Monster

If the length of the Loch Ness Monster is 20 meters and half of its length, how long is it?

2.13 Problem 13: Price of the Ball

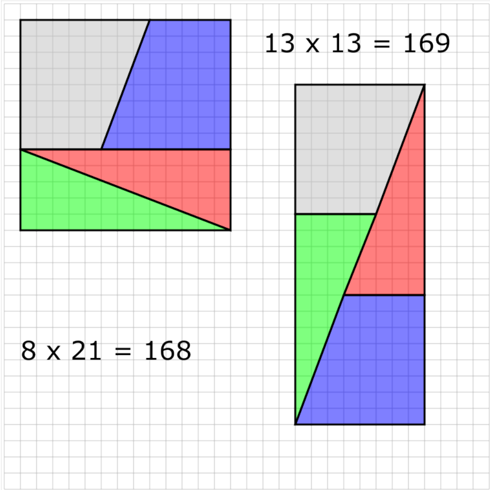
A golf club together with a ball costs 110 €. The price difference between a golf club and a ball is 10 €. What is the price of a ball?

2.14 Problem 14: Postman Problem

A postman distributes all letters in 45 minutes. If his assistant helps him, it takes only 20 minutes. How long would the assistant alone take?

2.15 Problem 15: One Unit Missing

The four forms in the 13×13 squares (left part) are apparently arranged in a rectangle of 8×21. Where is the missing unit? The square has $13 \times 13 = 169$ unit cells, but the rectangle only has $8 \times 21 = 168$ unit cells. Also find the hidden structure in this problem and generalize the results.



2.16 Problem 16: How old am I?

My age is a third of my father's age, but in 10 years my father will only be two times as old as I will be. What is my age today?

2.17 Problem 17: 7-Digits Puzzle (repeat)

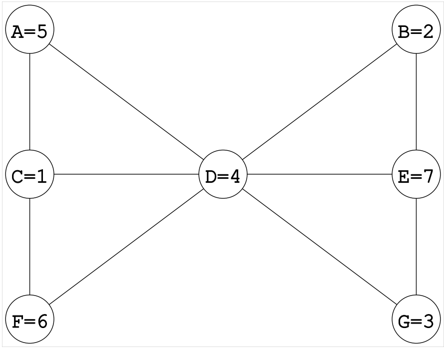
This is the same as the Puzzle 1 above: Use each digit from 1 to 7 exactly once, and place them into the circles of the Figure in such a way that the sum along each of the five lines is the same. This time, create a mathematical model with variables and constraints to solve the problem.

2.18 Problem 18: The Clock Puzzle (repeat)

This is the same as Problem 2 above: Divide the clock with a straight cut into two parts such that the sum of the numbers in both parts are equal? This time, create a mathematical model with variables and constraints to solve the problem.

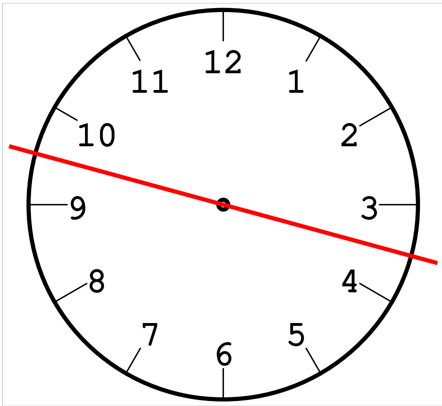
3 The Solutions

3.1 Problem 1: Seven Digits Puzzle



Loud thinking: "There are 7 numbers and 7 circles, this fact fits after all (If they were 8 circles and 7 numbers then this problem would not have a solution)! A straight line consists of three circles (hence three numbers). The figure shows some symmetry. The center circle seem to have a important position, because three lines cross it. Hmm... three number pairs, namely 1 + 7, 2 + 6, and 3 + 5 sum up to 8 each. What if 4 is in the center? Of course!" The rest is easy: no many possibilities remain. When modeling repeat the facts over and over again in different words. Write them down.

3.2 Problem 2: The Clock Puzzle



Loud thinking: "If one side must have the same sum than the other, then it is unlikely that all large numbers are on one side and all small numbers are on the other side. A mixture is needed. What about putting the largest and the smallest together? $12 + 1 = 13$. The same sum is given by $11+2 = 13$. Well, all 12 numbers can be paired to give 13. There are 6 pairings. Put three of them on one side and the other three on the other side!"

To prove that this is the unique solution, we must show that there exists no other consecutive sequence of numbers on the clock face that sums to the half of all numbers. Let's choose two unknown numbers x and y such that $1 \leq x < y \leq 11$. The sum of all numbers on the face from $y + 1$ to x in clockwise direction is the sum from $y + 1$ to 12 plus the sum from 1 to x. This is: the sum of all numbers from 1 to 12 minus the sum of all numbers from 1 to y plus the sum of all numbers from 1 to x:[3]

[3]Note that the sum of the numbers from 1 to n is given by $n(n + 1)/2$, (see Problem 7 below).

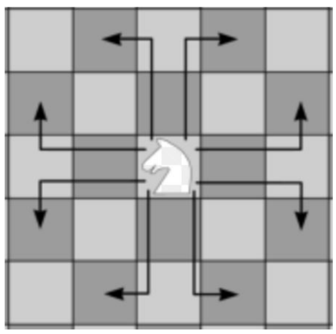
$$\left(\frac{12 \cdot 13}{2} - \frac{y(y+1)}{2}\right) + \frac{x(x+1)}{2}$$

This quantity must be half of all numbers from 1 to 12, that is, it must be $\frac{12 \cdot 13}{4}$. Simplifying the expression leads to:

$$(y-x) \cdot (y+x+1) = 6 \cdot 13$$

As $y-x < y+x+1$, we conclude that $y+x+1 = 13$ and $y-x = 6$. Resolving for x and y gives: $y = 9$ and $x = 3$. Hence the unique solution is a consecutive sequence from 10 to 3 in clockwise direction.

3.3 Problem 3: The 6-Knights Puzzle



This problem is a instructive example for the process of abstraction. In an initial step, the problem is reduced to its essential components. First we recall the rules of jumping for a single knight (see Figure at the right). Then the cells of the board are numbered from 1 to 12 to identify them.

The board and the knights do not matters, what only matters is from which number (cell) one can jump to another number: link these numbers with a straight line. The result is a well-known mathematical structure (a graph), consisting of lines connected to two locations (the numbers), see Figure 2 at the right. Note that from a knight board we have arrived at a very different structure (namely the numbers linked by straight lines), the problem has been reduced to the essential elements.

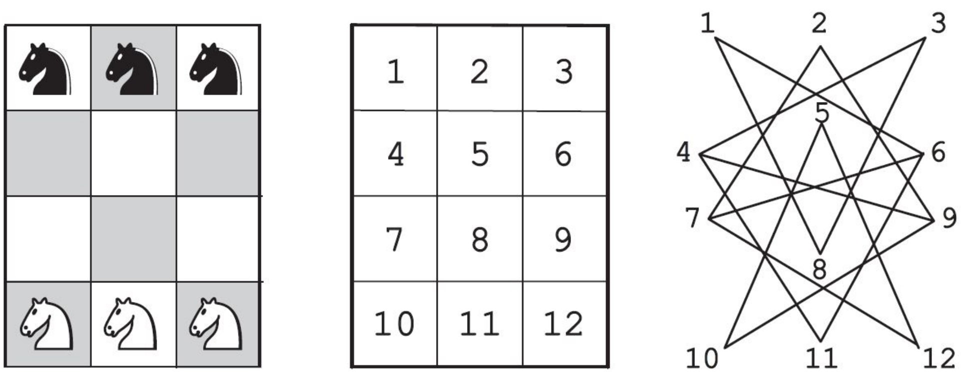


Figure 2: Transform the Problem to a Graph [4]

The positions of the numbers do not matter either, so we can "unfold" the graph and place the number wherever we want. However, the connecting lines must be preserved. Finally, we get a much simpler graph (see Figure 3 at the right), from which the solution can immediately be constructed: First the two knights in the middle are moved clockwise, till they have exchanged their places, then the outer knights are moved clockwise in the same way, each one move after the other.

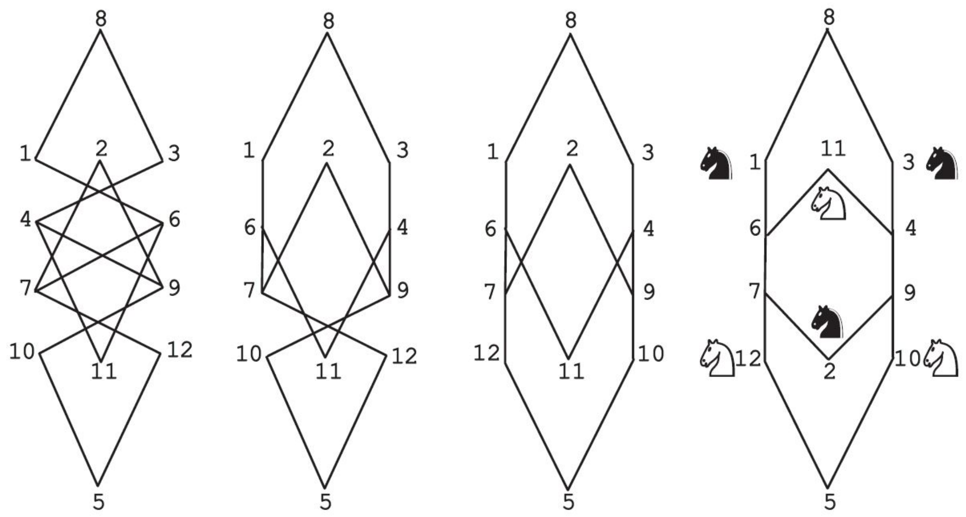


Figure 3: Simplifying a Graph [4]

Finally, the original problem can be reconstructed by executing the knights jumping in exactly that order. What is interesting about this puzzle – that is important for most real modeling problems – is the fact that one can ask (and answer very easily) "what of questions". In the solution above, for instance, knight 1 and knight 10 exchange their places. "What if" I want to exchange knight 1 with knight 13 instead? Is this possible? The reader would easily find a solution.

3.4 Problem 4: The 3-jug Problem

This is another example for using the concept of graph to model the problem. The problem can be formulated as a number of *states* and *transitions* between the states. A state is a particular filling of the jugs, for instance, a state is: "the 8-liter jug contains 8 liters of water, the 5-liter jug contains nothing, and the 3-liter jug contains nothing." Each state can be represented by a triple of numbers: (x, y, z), where x is the content of the 8-liter jug, y is the content of the 5-liter jug, and z is the content of the 3-liter jug. So, $(8, 0, 0)$ is the example just used before. We want to reach the state $(4, 4, 0)$. The capacities of the jug can also be represented by a triple: $(8, 5, 3)$.
The first step is to enumerated all states. There are 216 potential states, namely, all states (x, y, z) with $0 \leq x \leq 8$, $0 \leq y \leq 5$, and $0 \leq z \leq 3$, which is $9 \cdot 6 \cdot 4 = 216$. However, only states with the condition $x + y + z = 8$ are allowed (no water should be added or removed). Furthermore, at least one jug must be full or empty – following the rules of pouring. This condition can be represented as a Boolean expression. $x = 0 \lor x = 8 \lor y = 0 \lor y = 5 \lor z = 0 \lor z = 3$.[4]
There are 16 remaining possible states, they are:

$\{(0, 5, 3), (1, 4, 3), (1, 5, 2), (2, 3, 3), (2, 5, 1), (3, 2, 3), (3, 5, 0), (4, 1, 3),$

[4] $\lor$ is the Boolean "or" operator.

$(4,4,0),(5,0,3),(5,3,0),(6,0,2),(6,2,0),(7,0,1),(7,1,0),(8,0,0)\}$

The basic operation is to pour water from one jug A to another jug B in a way that either A is emptied or B is filled. We are looking for the shortest sequence of operations that reaches the state $(4,4,0)$ starting with state $(8,0,0)$. Such a basic operation is called a *transition from one state to another*. You noticed? The "problem of the 3-jugs" has been transformed into a "problem of states and transitions between the states". The final step is to position the states somewhere in the plane and link two states with an arrow, if a transition from the first to the second state is possible. The result is a (directed) graph, a graph with directed links. The problem now is reduced to find the shortest (direct) path[5] in this graph from state $(8,0,0)$ to the state $(4,4,0)$. The resulting graph and the shortest path in red is given in Figure 4.
I guess it is clear how to interpret the solution: In the first step, take the 8-liter jug and fill the 5-liter jug, the second step is to take the 5-liter jug and to fill the 3-liter jug, the thrid step is to empty the 3-liter jug and pour it to the 8-liter jug, and so on, 7 steps are needed to get to the goal $(4,40)$.

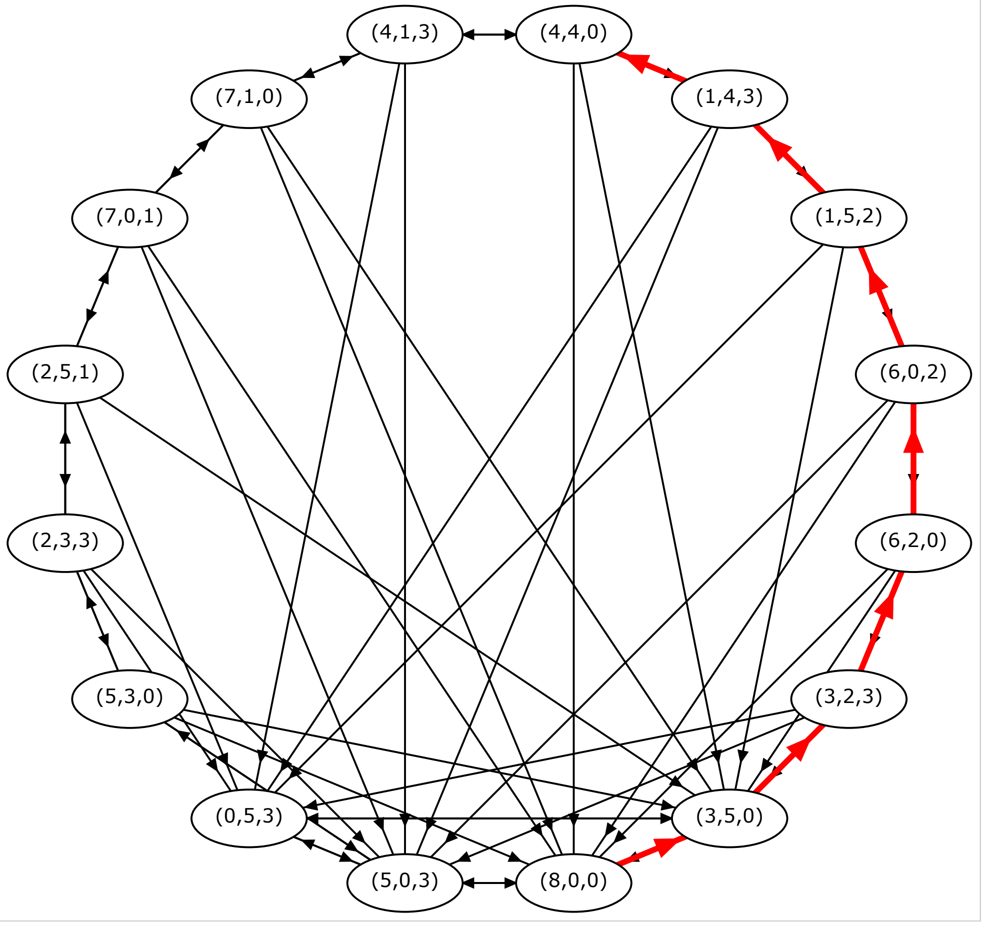


Figure 4: Solution of the 3-Jug Problem

Interestingly, we not only know now how to solve this particular 3-jug problem but all kind of jug problems with other capacities and with a different number of jugs. This is a strong indication of a good model: if the model's vocabulary

[5]Finding the shortest path in a graph is a well-known problem in operations research.

stimulates others undiscovered aspects of the problem or guides the research to similar problems, then the model might be reasonably good. If, however, the vocabulary is "sticky" or "artificial" then the model is probably not very useful.

3.5 Problem 5: The Fake Coin Puzzle I

Two weighings are sufficient! That is astonishing. First, the coins are numbered from 1 to 8. Of course, the first idea would be to weight all coins: 4 on one side of the balance and 4 on the other side. The fake coin is then at the lighter side, this is repeated with the four on the lighter side, etc. This method would lead to 3 weighings. But what if we only weight 6 coins initially? Coins 1,2,3 on one side of the balance and 4,5,6 on the other side? Because then we know: if the left side is lighter, the fake coin is there, if the right side is lighter, then it is on the right side, and if they are equal, then the face coin must be 7 or 8. Bingo! That is the key idea: to separate the coins into three groups. This key idea is repeated in the second weighing. Suppose the left side with the coins 1, 2, 3 is lighter, then we know the fake coin is there. Now we weight only coin 1 against coin 2. If the left is lighter then the fake coin is 1, of the right is lighter then the fake coin is 2, otherwise the fake coin is 3, which was not weighted! The whole weighings process is best shown in a Figure 5.

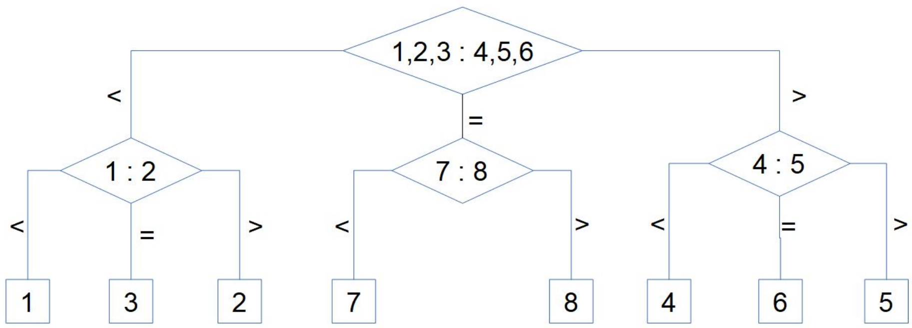


Figure 5: 8-coins Puzzle Solved

3.6 Problem 6: The Fake Coin Puzzle II

3 weighings are sufficient! This problem is an instructive example for *similarity*. Is this problem not similar to the previous one? Of course, it is! Can we use similar ideas? Let's try to apply the ideas from the the previous problem! Again, the coins are numbered from 1 to 12. First, we introduce a notation to represent the actual knowledge about a coin: the symbols $+$, $-$, and $\pm$ following the coin number mean that the coin is "lighter", "heavier" or "don't know", for example "1+" means we know that the first coin is heavier if it is the fake coin. Similar to the previous problem, the first 4 coins are put on the left side of the balance and the next 4 coins on the right side, and 4 other coins are not touched. After the first weighing we know: if the left side is lighter, then the fake coin is within the first 4 coins and it is lighter *or* the fake coin is within

the next 4 coins (coin 5 to 8) and it is heavier; if the two sides are equal, we can forget about all 8 coins and the fake coin is within the last 4 coins (not on the balance). We repeat the same idea, the weighings are shown in Figure 6 shows the solution.

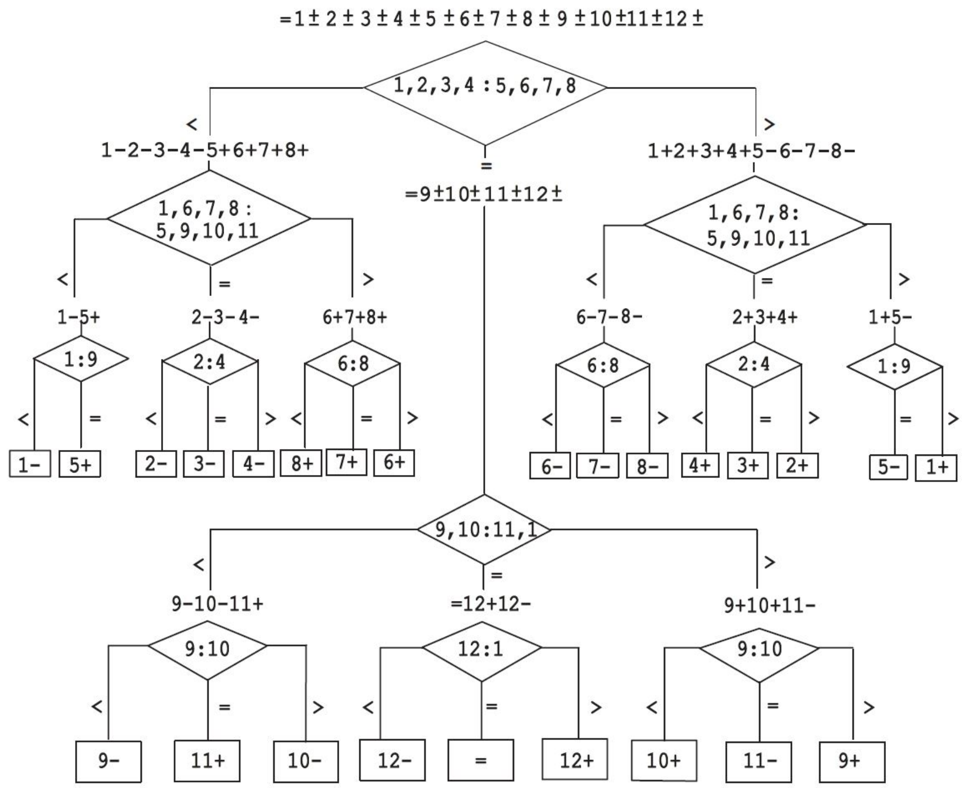


Figure 6: 12-coins Puzzle Solved [4]

Modeling by similarity is very common. One of the first question with a new problem is: "Do I know a similar problem?" Often the problem is different in many respects, but some key ideas can be reused.

3.7 Problem 7: The Horse Race Puzzle

This problem is sometimes asked in a technical interview question at companies like Google (better be prepared!) (Youtube Video). The problem can be solved by a systematical ordering of the facts and a clear grouping and regrouping of objects, no more is needed – these are other important tasks in modeling in general. *Order*, *sort*, *collect*, and *name* the components before doing more creative things in modeling. Often, the elements "fall into the right place" just by naming them systematically. In mathematical modeling the key element is often to identify the right *variables*.
Let's analyze the horse problem: Only 5 horses can race at the same time, so:

1. Group the 25 horses into 5 groups of 5 horses. The groups are named *a*, *b*, *c*, *d*, *e*. Mark each horse with its group name. Hence, 5 horses are marked with an "a", 5 with a "b", and so on.

2. Let each group race, that is, the five horses in group *a* do the first race, then the five horses in group *b*, etc. **This gives 5 races**.

3. Mark each horse with its rang. Hence, the fastest horse in each group is marked with the number 1, the second fastest with number 2, etc. Each horse is now marked with its group name an its rank after the first five races.

4. Let the 5 horses with number 1 do another race (**race number 6**).

5. Order the 5 groups according to the fastest horses in that last race. So, for instance, suppose $c1$ is the fastest horse, second is $b1$, third is $a1$, fourth is $d1$, and the slowest is $e1$ in this 6th race, then order the groups from top to bottom accordingly as shown in Figure 7 (first row contains all horses named c; second row all horses named b, etc.).

c5	c4	c3	c2	c1
b5	b4	b3	b2	b1
a5	a4	a3	a2	a1
d5	d4	d3	d2	d1
e5	e4	e3	e2	e1

Figure 7: Ordering of the Groups

6. Clearly, the horse $c1$ is the fastest overall. Why? It is faster than any a horse (from the first race), and it is faster than any other 1 horse, and any 1 horse was faster than any 2 horse etc.

7. Who is second and third? In the group of the 1 horses only $b1$ and $c1$ can be second or third (but not $d1$ or $e1$). In group a only $a1$ could be third (all others in group a were slower). In group c and b the two fastest (beside $c1$) can be second or third. So let race the horses $c2$, $c3$, $b1$, $b2$, and $a1$ a final time (**race number 7**). The two fastest horses in that last race are second and third.

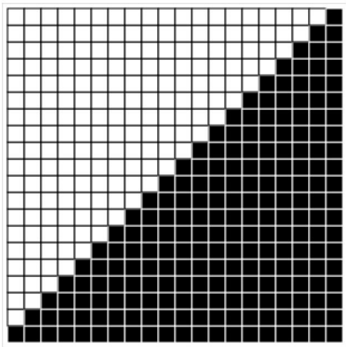
c5	c4	c3	c2	c1
b5	b4	b3	b2	b1
a5	a4	a3	a2	a1
d5	d4	d3	d2	d1
e5	e4	e3	e2	e1

Figure 8: Ordering of the Groups

Another important concept in modeling appears here: We simplify by leaving out the unessential, but we simplify also by *implicitly assuming certain conditions*. In our example of the horse racing, we suppose that the horses perform in the same way *in each race*, that is, the fastest in the first race is also the fastest in the 6th race, for instance. In all real problems, this is an *idealized* assumption. We know that this is not true. So, is the model then useless? Well, it depends on the context. In our horse-race example we may *define* the horse to be the fastest if it is the fastest in the 6th race, but it might have been slower than any other 1 horse in the 5 first races (when measured by a watch). In any case, it is important to be aware of all hidden assumptions of a model and in what context it would be inappropriate to apply the model.

3.8 Problem 8: Adding Numbers

$1 + 2 = 3$, $3 + 3 = 6$, $6 + 4 = 10$, $10 + 5 = 15$,... I guess this is a too lengthy calculation. What if we pile small black squares, first one (on the top), then two, then three from top to bottom, etc. like in the Figure. Then we complete the piles with white squares to a larger square (of 20×20 in the Figure).



Now the problem can be reduced to the question: How many black squares are there? We have 20×20 total squares in this example, half of them plus half of 20 are black. Why? Look carefully, there are more black than white cells, because the diagonal also contains only black cells. If we made half of the diagonal (that is 10) white, then we would have exactly the same number of white and black cells. Hence, the number of black cells is: $20 \times 20/2 + 20/2 = 210$.

The same reasoning holds if there is a 1000×1000 square: Half of 1000^2 plus $1000/2$ are black, or in a 3×3 square, where $3^2/2 + 3/2$ are black – but wait a minute: $3/2$ is not an integer number. Well, $3^2/2$ is not integer either, but together they are integer (draw it, to verify). In general, for an $n \times n$ large square, the number of black squares – that is the sum from 1 to n) – therefore is:

$$\frac{n^2}{2} + \frac{n}{2} = \frac{(n+1)n}{2}$$

This approach shows two important concepts in modeling: (1) Transform the problem into a very different (graphical) problem (piles of squares) that is structurally identical, (2) try first to pile 3, then 4, etc. Can you see the pattern? Can it be generalized?

A different approach is with arranging numbers. Try this in Figure 9. How many 1001 do we have? 1000! Since each number is counted twice, we get: $1001 \cdot 1000/2$, which confirms the formula above.

A digression: Proof that the formula is true for all $n > 0$. Proof by induction:

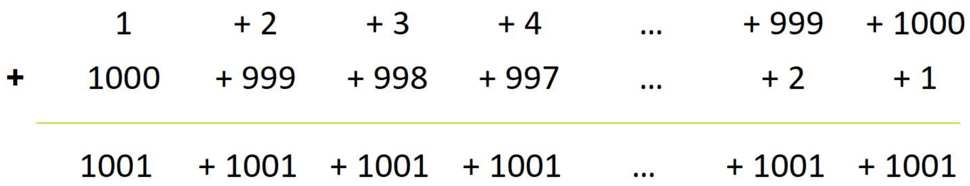


Figure 9: Two Lines of Numbers

(1) prove that it is true for $n = 1$. This is the case, since:

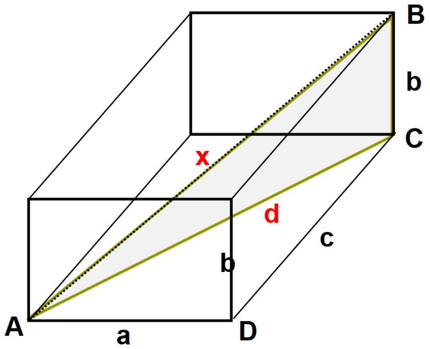
$$\sum_{i=1}^{1} i = \frac{(n+1)n}{2} = \frac{2 \cdot 1}{2} = 1$$

(2) prove that if it is true for n then it is true for $n+1$:

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + n + 1 = \frac{(n+1)n}{2} + n + 1 = \frac{n^2 + 3n + 2}{2} = \frac{(n+2)(n+1)}{2}$$

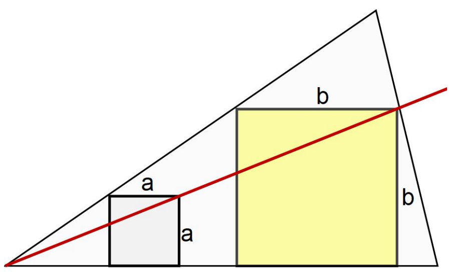
3.9 Problem 9: 3d-Diagonal

This problem shows another important concept in modeling: Reduce the problem to another known problem (Pythagoras). Add a line from A to C, then the triangle ABC form an auxiliary right-angled triangle where $x^2 = d^2 + b^2$. ADC forms another triangle for which holds: $d^2 = a^2 + c^2$. Eliminating d gives: $x = \sqrt{a^2 + b^2 + c^2}$.



3.10 Problem 10: Square Inside a Triangle

If you cannot see how to begin, just draw a square inside the triangle that shares a side with the triangle and touches one other side of the triangle. By drawing the red line, the solution pops up immediately. The key concept here is proportionality.



3.11 Problem 11: Birthday Paradox

For this problem, it is easier to calculate the probability that 2 persons do *not* have the same birthday.

- With 2 persons, the probability that they do not have the same birthday is $\frac{364}{365}$.

- With 3 persons the probability, that 2 persons do not have the same birthday, is $\frac{364}{365} \cdot \frac{363}{365}$.
- With 4 persons the probability, that 2 persons do not have the same birthday, is $\frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365}$. Do you see the pattern now?

Within a group of n persons the probability, that 2 persons do not have the same birthday, is therefore:

$$P'(n) = \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{365-n+1}{365}$$

This formula can be transformed to:

$$P'(n) = \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdot \ \ldots \ \cdot \left(1 - \frac{n-1}{365}\right)$$

The initial problem that 2 persons out of n persons *do have* the same birthday is then $P(n) = 1 - P'(n)$. Figure 10 is a graph of this function $P(n)$. The graph shown that the probability that 2 persons in a group of 30 persons have the same birthday is more than 70%. In a party with 23 persons the probability is already larger than 50%.

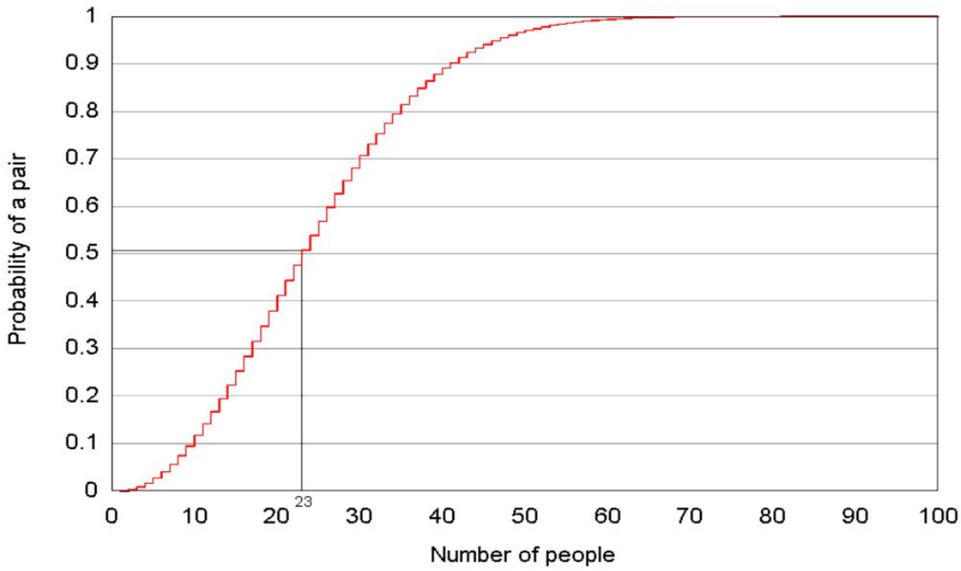


Figure 10: Probability $P(n)$

The birthday paradox problem has a very useful application in computer science for calculating the probability that 2 entries clashes in a hash table: Given a hash table of size 365, what is the probability that 2 entries clashes on the same memory location after n entries. This is a good example for the fact that the same model can have very different *interpretations* (or applications) in completely different domains.
Another aspect makes this problem interesting from a modeling point of view. The calculation of the expression for $P'(n)$ above is laborious. Isn't there a simpler model? Indeed we know that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad (\text{ and for } x \ll 1, e^x \approx 1 + x)$$

Let $x = -1/365$, then $x = -2/365$, ... then $P'(n)$ can be expressed as:

$$P'(n) = e^{-1/365} \cdot e^{-2/365} \cdot \ \ldots \cdot e^{(1-n)/365}$$
$$= e^{-(1+2+\ldots+(n-1))/365} = e^{(n^2-n)/730}$$

Simplifying and replacing by another approximation gives:

$$P(n) \approx 1 - e^{n^2/730}$$

Hence, the birthday problem can be approximated by a much simple model that is a very good approximation. One can easily calculate the probabilities using a hand calculator.

3.12 Problem 12: Length of Loch Ness Monster

30 meters is *not* correct! Let it do carefully. Suppose the length of the Loch Ness Monster is x meters, then this is the same as 20 meters plus half of x. So:

$$x = 20 + x/2 \quad , \qquad \text{and } x = 40$$

The Loch Ness Monster is 40 meters long.

3.13 Problem 13: Price of the Ball

Things are not as they seems to be at first glance, sit down and calculate! Let x be the price of a golf club, and let y be the price of the ball. Together they cost 110, hence $x + y = 110$. The difference is 10, that means: $x - y = 10$. Now its is easy to see that $x = 105$ and $y = 5$. The price of the ball is 5 €(and not 10 €).

3.14 Problem 14: Postman Problem

What is the variable? "The time that the assistant alone takes in minutes", name this quantity: x. It seems clear: The *difference* between "both doing the job" and "the postman alone does the job" is 25, hence, $x = 25$. Verify! Suppose we would ask for the postman's time y alone given the assistant time 25 and together they take again 20. Using a symmetrical argument, we could then say that: $25-y = 20$, or $y = 5$ which contradicts the fact that the postman alone takes 45 mins and not 5 mins! Therefore, *this approach is completely wrong!*
Well then, "Together they take 20 minutes, then *on average* each alone takes 40 minutes." Since the postman really takes 45 minutes – he is slower then the average – the assistant must be faster then average, namely by $45-40 = 5$ minutes faster, hence: $x = 35$. This is again not correct! Why? Suppose together they would take only 10 minutes, then the average is 20 and the difference between the postman (45 minutes) and the average is 25. That means that the assistant would take $20-25 = -5$ minutes. That cannot be! Whatever short time they take together, the time of the assistant must *always* be positive. Therefore, *the approach is wrong again!*

We have to look at the problem from a *different angle*. So, let's concentrate on the *number of letters distributed*, and not on the number of minutes. The right question here is: How many letters are distributed in a given amount of time by (1) the postman, (2) the assistant, (3) together. Surely, together they distribute the sum of the letters of each individually (supposing each works the same speed together or not together), that is, number of letters the postman distributes plus the number of letters the assistant distributes must be the numbers of the letters both distribute together in a given amount of time. Hence, we look at the number of letters distributed in one minute:

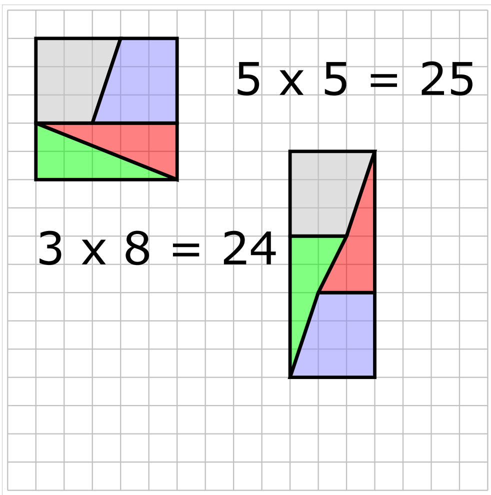
- In one minute, the postman distributes $1/45$ of all letters.
- In one minute, the assistant distributes $1/x$ of all letters.
- In one minute, together they distribute $1/20$ of all letters.

Now the model is easy to derive and to find our x, it is:

$$\frac{1}{45} + \frac{1}{x} = \frac{1}{20}$$

Resolving the equation gives $x = 36$. The assistant alone takes 36 minutes. Suppose, as an extreme case, together they would take only 1 minute, then the assistant alone would take $45/44$ minutes (calculate!). This number is slightly higher than 1, which makes sense, because they together *always* do it in less time.
Well! One could argue that together they will chat a lot and wasting time or stop for a bite. This may happen, but then we have *another problem*. We made the silent assumption that their performance do not change when working together or working alone.

3.15 Problem 15: One Unit Missing



We take a smaller instance for the same problem (see Figure at the right). Now one can see immediately where the problem is: the diagonal in the 3×8 rectangle ist *not* a straight line. Hence, none of the forms is a triangle. A quick calculation also shows that they *cannot* be triangles. The proportion of the height and the length of these triangles is $\frac{3}{8}$. At the position 5 at the length, the height of this "triangle" is 2. But we know that $\frac{3}{8} \neq \frac{2}{5}$. Hence, these forms are *not* triangles.

How did we discover these figures? They are "special numbers" 3×8 is almost 5×5, and 8×21 is almost 13×13. Sounds familiar? 1, 1, 2, 3, **5, 8, 13, 21,** 34,... These are consecutive Fibonacci numbers. These numbers are defined by:

$$F_1 = 1\ , \quad F_2 = 1\ , \quad F_n = F_{n-1} + F_{n-2}\ , \qquad \text{with } n \geq 2$$

The Fibonacci numbers have many interesting properties, one of them is

$$F_{n+1} \cdot F_{n-1} - F_n^2 = (-1)^n\ , \qquad \text{with } n \geq 1$$

Proof (by induction):

1. The property is true for $n = 1$, since $1 \cdot 0 - 1^2 = (-1)^1$.

2. Supposing the property is valid for n, then it is valid for $n + 1$, namely we substitute F_{n-1} with $F_{n+1} - F_n$ in the property and we get:

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$$

   By multiplying with -1 and transforming it we finally get:

$$F_{n+2} \cdot F_n - F_{n+1}^2 = (-1)^{n+1}$$

   That proves the property.

3. In particular, for $n = 6$ we have: $13 \cdot 5 - 8^2 = (-1)^6$ producing the initial graph of the problem. We also have: $8 \cdot 3 - 5^2 = (-1)^5$.

3.16 Problem 16: How old am I?

Let us translate this problem into a mathematical formulation step by step, to remember the modeling process in such cases (see [1]) :

1. *Understanding the problem*: What are we looking for? What do we known? What are the conditions? Are the data sufficient, are some data contradictory, irrelevant or redundant? Draw a figure! Introduce a suitable notation. Write it down. If you are stuck: begin again.

2. *Design a plan*: Do you know a similar problem? Find the connection between the data and the unknowns! Maybe you need to design auxiliary problems and intermediary steps! Look at the unknowns! Go back to the definitions! Decompose the problems and solve the parts! Did you use all data? Did you use all conditions?

3. *Carry out a plan*: Write it down step by step! Can you show that each step is correct? Make plausibility tests in each step. If possible simplify and do the calculations.

4. *Looking back*: Examine the solution. Can you check the result or the argument? Can you derive the result differently? Does the result make sense? Why? Why not?

Let's look at our problem now :

1. What are we looking for in our problem? “my age today”! This is unknown, so let us introduce the symbol a for “my age today”. The symbol a stands for a positive number.

2. In the same way, let us introduce the symbol b for “my father’s age today”, the symbol c for “my age in ten year”, and the symbol d for “my father’s age in ten years”.

3. Draw a figure (see Figure 11)

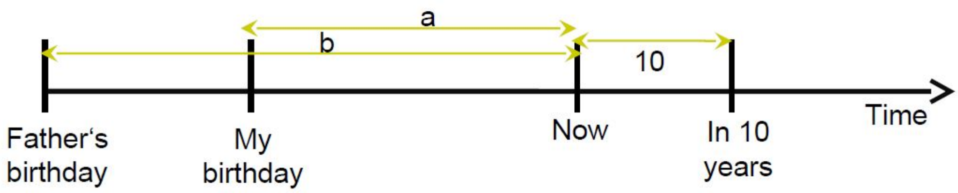


Figure 11: A Figure for the Problem

4. Connect the data with the unknown. It is clear from the statement that

$$c = a + 10 \quad \text{and} \quad d = b + 10$$

5. Furthermore: “My age is a third of my father’s age” means that

$$a = b/3$$

6. Finally: “In 10 years my father will only be two times as old as I will be” gives:

$$d = 2c$$

7. Did we use all data and all conditions? I guess, yes!

8. We wrote down the complete problem in mathematics. Simplify it: eliminate the variables c and d gives the following:

$$a = b/3 \qquad \text{and} \qquad b + 10 = 2(a + 10)$$

9. Calculating gives: $a = 10$ and $b = 30$.

10. Looking back: My age is 10, in ten years I am 20, my father’s age is 30, and in ten years he is 40. My father is three times older. Correct! But in ten years he is 40 and I am 20, so he is only two times older. Also correct! Everything is matching. So we are done.

3.17 Problem 17: 7 Digits Puzzle (repeat)

The 7-digits puzzle is a good brain game. Interestingly, it could be formulated as a mathematical model. So how would one model this problem? First question: what are the variables? Answer: Which number to place in which circle. First of all, all the numbers must be different from each other and in the range $[1, 7]$. Hence, seven integer variables are introduced: x_i with $i \in \{1, \dots, 7\}$ for the 7 circles. For instance, x_1 is the number in circle A, x_2 is the number in circle B, and so on. Clearly, the three number in a line must be equal. There are five lines, so the constraints are as follows:

$$
\begin{aligned}
& x_2 + x_4 + x_6 = x_1 + x_4 + x_7 \\
& x_3 + x_4 + x_5 = x_1 + x_4 + x_7 \\
& x_2 + x_5 + x_7 = x_1 + x_4 + x_7 \\
& x_1 + x_3 + x_6 = x_1 + x_4 + x_7 \\
& x_i \in \{1, \dots, 7\}, \text{ all distinct}
\end{aligned}
$$

The model was implemented in the modeling language LPL and can be executed here: Puzzle7.

3.18 Problem 18: The Clock Puzzle (repeat)

This problem can also be formulated as a mathematical model. We want to known for each number on the clock's face which is on one side and which is on the other side. For this kind of situation, $0 - 1$ variables are used – that is, variables that have only the value "zero" or "one" (or *true* and $false$). Let's introduce 12 binary variable x_i with $i \in \{1, \dots, 12\}$, with the following meaning: $x_i = 1$ if the number i is on one side (say side A), and $x_i = 0$ if the number i is on the other side (say side B). We have two conditions: (1) The sum of the numbers on side A must be half of all numbers. (2) Exactly two pairs of adjacent numbers must be on different sides. Suppose, one pair of numbers on different sides is 9 and 10. Then this means that x_9 and x_{10} must have different values, if one is 0 the other must be 1 and vice versa, (hence $x_9 \neq x_{10}$ would be a sufficient condition). This must be the case for *exactly two pairs* of numbers (The operator "exactly$(2)_i$ (E_i)" the model below means: "exactly 2 out of all i expressions E_i are true"). Hence, the mathematical model for this problem is:

$$
\begin{aligned}
& \sum_i i \cdot x_i = \left(\sum_i i \right) / 2 \\
& \text{exactly}(2)_i \ (x_i \neq x_{i \bmod 12+1}) \\
& x_i \in \{0, 1\}, \quad \text{forall } i \in \{1, \dots, 12\}
\end{aligned}
$$

The model was implemented in the modeling language LPL and can be executed here: DrawClock.

4 Conclusion

This paper gives a short introduction to modeling with examples of how to model. The examples show that modeling is not trivial. Although the problems presented here are quiet simple, so to say, find a good model formulation is not always easy. How much more difficult must it be to formulate real-life problems with a lot of data and constraints. Modeling an art, sure, but a art that can be learnt: one must practise. Continue modeling, look up my books with a lot of examples see My Modeling Books.

References

[1] Polya G. *How To Solve It*. Penguin Books 1990, London, 1957.

[2] Polya G. *Mathematical Discovery, On Understanding, Learning, and Teaching Problem Solving*. John Wiley, Combined Edition, 1968.

[3] Dudeney H.E. *Amusements in Mathematics*. Thomas Nelson and Sons, 1917.

[4] Levitin Maria Levitin A. *Algorithmic Puzzles*. Oxford University Press, 2011.

[5] MatMod. Homepage for Learning Mathematical Modeling : `https://matmod.ch`.

[6] Hürlimann T. Various Modeling Tools. `https://matmod.ch/lpl/doc/modeling5.pdf`.

[7] Hürlimann T. What is modeling ? `https://matmod.ch/lpl/doc/modeling1.pdf`.

[8] Kuhn T.S. *The Structure of Scientific Revolutions*. University of Chigago Press, 1962.