What is “Modeling” ?

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Abstract

Whereas mathematical solution and visualization techniques are omnipresent, modeling techniques – translating a problem into formal notation – received only little attention, although modeling is one of the most creative and mostly quite challenging task. Is is an art and a discipline on its own right, and deserves a outstanding place when learning how to solve problems.

This paper is the first of several papers on modeling and it gives an informal and a formal introduction to the concepts of *modeling* and *model*. It explains the importance and functions of models. Certainly, “model” is a very general term, an its meaning varies throughout various branches of science (**Science is modeling**). Since we are interested basically in *mathematical models*, a precise definition is given. An intuitive notion is given in this paper of some basic concepts, such as *Boolean and arithmetic operators*, *sets*, *vectors*, *tuples*, *vector spaces*. However, to understand fully the mathematical examples, the reader should be familiar with these concepts. More is not needed at this point. Many more mathematical models can be found in my books, especially my puzzle book: See My Modeling Books.

1 Introduction

Observation

Observation is the ultimate basis for our understanding of the world around us. But observation alone only gives information about particular events; it provides little help for dealing with new situations. Our ability and aptitude to recognize similarities and patterns in different events, to distil the important factors for a specific purpose, and to generalize our experience enables us to operate effectively in new environments. The result of this skill is *knowledge*, an essential resource for any intelligent agent.

Knowledge

Knowledge varies in sophistication from simple classification to understanding and comes in the form of principles and models. A *principle* is simply a general assertion and is expressed in a variety of ways ranging from saws, slogans, opinions to mathematical equations. They can vary in their validity and their precision. A *model* is, roughly speaking, an analogy or a mapping for a certain object, process, or phenomenon of interest. It is used to explain, to predict, or to control an event or a process. For example, a miniature replica of a car, placed in a wind tunnel, allows us to predict the air resistance or air eddy of a real car; a globe of the world allows us to estimate distances between locations; a graph consisting of nodes (corresponding to locations) and edges (corresponding to streets between the locations) enables us to find the shortest path between any two locations without actually driving between them; and a differential equation system enables us to balance an inverted pendulum by calculating at short intervals the speed and direction of the car on which the pendulum is fixed.

Model = representation of knowledge

A model is a powerful means of structuring knowledge, of presenting information in an easily assimilated and concise form, of providing a convenient method for performing certain computations, of investigating and predicting new events. The ultimate goal is to make decisions, to control our environment, to predict events, or just to explain a phenomenon.

1.1 Models and their Functions

Models can be classified in several ways. Their characteristics vary according to different dimensions: function, explicitness, relevance, formalization. They are used in scientific theories or in a more pragmatic context. Here are some examples classified by function, but also varying in other aspects. They illustrate the countless multitude of models and their importance in our life.

Models can explain phenomena

Models can *explain phenomena*. Einstein's special relativity explains the Michelson-Morley experiment of 1887 in a marvelously simple way and overruled the ether model in physics. Economists introduced the IS-LM or rational expectation models to describe a macro-economical equilibrium. Biologists build mathematical growth models to explain and describe the development of populations. Modern cosmologists use the big-bang model to explain the origin of our universe, etc.

Models control our environment

There are also models *to control our environment*. A human operator, e.g., controls the heat process in a kiln by opening and closing several valves. He or she knows how to do this thanks to a learned pattern (model); this pattern could be formulated as a list of instructions as follows: "IF the flame is bluish at the entry port, THEN open valve 34 slightly". The model is not normally explicitly described, but it was learnt implicitly from another operator and maybe improved, through trial and error, by the operator herself. The resulting experience and know-how is sometimes difficult to put into words; it is a kind of tacit knowledge. Nevertheless one could say that the operator acts on the basis of a model she has in mind.

Implicit models

Explicit models

On the other hand, the procedure for aeroplane maintenance, according to a detailed checklist, is thoroughly *explicit*. The model is possibly a huge guide that instructs the maintenance staff on how to proceed in each and every situation.

Chemical processes can be controlled and explained using complex mathematical models. They often contain a set of differential equations which are difficult to solve. These models are also explicit and written in *a formalized language*.

Other models are used to control a social environment and often contain *normative* components. Brokers often try, with more or less success, to use guidelines and principles such as: "the FED publicizes a high government deficit provision, the dollar will come under pressure, so sell immediately". Such guidelines often don't have their roots in a sophisticated economic theory; they just prove true because many follow them. Many models in social processes are of that type. We all follow certain principles, rules, standards, or maxims which control or influence our behavior.

Models for making decisions

Still other models constitute the basis *for making decisions*. The famous waterfall model in the software development cycle says that the implementation of a new software has to proceed in stages: analysis, specification, implementation, installation and maintenance. It gives software developers a general idea of how to proceed when writing complex software and offers a rudimentary tool to help them decide in which order the tasks should be done. It does not say anything about how long the software team have to remain at any given stage, nor what they should do if earlier tasks have to be revised: It represents *a rule-of-thumb*.

An example of a more *formal* and complex decision-making-model would be a mathematical production model consisting typically of thousands of constraints and variables as used in the petroleum industry to decide how and in what quantities to transform crude oil into petrol and fuel. The constraints – written as mathematical equations (or inequalities) – are the capacity limitations, the availability of raw materials, etc. The variables are the unknown quantities of the various intermediate and end products to be produced. The goal is to assign numerical values to the variables so that the cost are minimized, profit is maximized or some other goals are attained.

Both models are tools in the hand of an intelligent agent and guide her in her activities and support him in his decisions. The two models are very different in form and expression; the waterfall model contains only an informal list of actions to be taken, the production model, on the other hand, is a highly sophisticated mathematical model with thousands of variables which needs to be solved by a computer. But the degree of formality or complexity is not necessarily an indication of the "usefulness" of the model, although a more formal model should normally be more precise, more concise, and more consistent. Verbal and pictorial models, on the other hand, give only a crude view of the real situation.

Models might also be misleading

Models may or may not be *pertinent* for some aspects of reality; they *may or may not correspond* to reality, which means that models can be misleading. The medieval model of human reproduction suggesting that babies develop from homunculi – fully developed bodies within the woman's womb – leads to the absurd conclusion that the human race would become extinct after a finite number of generations (unless there is an infinite number of homunculi nested within each other). The model of a flat, disk-shaped earth may have prevented many navigators from exploring the oceans beyond the nearby coastal regions because they were afraid of "falling off" at the edge of the earth disk. The model of the falling profit rate in Marx's economic theory predicted the self-destruction of capitalism, since the progress of productivity is reflected in a decreasing number of labor hours relative to the capital. According to this theory, labor is the only factor that adds plus-value to the products. Schumpeter agreed on Marx's prediction, but based his theory on a very different model: Capitalism will produce less and less innovative entrepreneurs who create profits! The last two examples show that very different sophisticated models can sometimes lead to the same conclusions.

In neurology, artificial neural networks, consisting of a connection weight matrix, could be used as models for the functioning of the brain. Of course, such a model abstracts from all aspects except the connectivity that takes place within the brain. However, some neurologists believe that only 5%(!) of information passes through the synapses. If this turned out to be true, artificial neural nets would indeed be inappropriate models for the functioning of the brain.

Models are everywhere

One can see from these examples that models are ubiquitous and omnipresent in our lives. "The whole history of man, even in his most non-scientific activities, shows that he is essentially a model-building animal" [5]. We live with "good" and "bad", with "correct" and "incorrect" models. They govern our behavior, our beliefs, and our understanding of the world around us. Essentially, we see the world by means of the models we have in mind. The value of a model can be measured by the degree to which it enables us to answer questions, to solve problems, and to make correct predictions. Better models allow us to make better decisions, and better decisions lead us to better adaptation and survival – the ultimate "goal" of every being.

1.2 The Advent of the Computer

Mathematical models

This paper is not about models in general, their variety of functions and characteristics. It is about a special class thereof: mathematical models. Mathematics has always played a fundamental role in representing and formulating our knowledge. As sciences advance they become increasingly mathematical. This tendency can be observed in all scientific areas irrespective of whether they are application- or theory-oriented. But it was not until last century that formal models were used in a systematic way to solve practical problems. Many problems were formulated mathematically long ago, of course. But often they failed to be solved because of the amount of calculation involved. The analysis of the problem – from a practical point of view at least – was usually limited to considering small and simple instances only.

Mathematical models and computers

The computer has radically changed this. Since a computer can calculate extremely rapidly, we are spurred on to cast problems in a form which they can manipulate and solve. This has led to a continuous and accelerated pressure to formalize our problems. The rapid and still ongoing development of computer technologies, the emergence of powerful user environment software for geometric modeling and other visualizations, and the development of numerical and algebraic manipulation on computers are the main factors in making modeling – and especially mathematical modeling – an accessible tool not only for the sciences but for industry and commerce as well.

Of course, this does not mean that by using the computer we can solve every problem – the computer has only pushed the limit between practically solvable and practically unsolvable ones a little bit further. The bulk of practical problems which we still cannot, and probably never will be able, to solve efficiently, even by using the most powerful parallel machine, is overwhelming. Whether quantum-computer will change this situation, will be seen in the future. Nevertheless, one can say that many of the models solved routinely today would not be thinkable without the computer. Almost all of the manipulation techniques in mathematics, currently taught in high-schools and universities, can now be executed both more quickly and more accurately on even cheap machines – this is true not only for arithmetic calculations, but also for algebraic manipulations, statistics and graphics. It is fairly clear that all of these manipulations are already standard tools on every desktop machine. Sixth years ago, the hand calculator replaced the slide rule and the logarithm tables, now the computer replaces most of those mathematical manipulations which we learnt in high-school and even at university.

1.3 New Scientific Branches Emerge

Operations Research

Some research communities such as *operations research* (OR) owe their very existence to the development of the computer. Their history is intrinsically linked to the development of methods applicable on a computer. The driving force behind this development in the late 1940s was the Air Force and their Project SCOOP. Numerical methods for linear programming (LP) were stimulated by two problems they had to solve: one was a diet problem: Its objective is to find the minimum cost of providing the daily requirement of nine nutrients for a soldier from a selection of seventy-seven different foods. Initially, the calculations were carried out by five computers[1] in 21 days using electromechanical desk calculators. The simplex method for linear programming (LP), discovered and developed by Dantzig in 1947 , was and still is one of the greatest successes in OR. Together with good presolve techniques we can solve today almost any LP problems with millions of variables and constraints in a fraction of time.

Artificial intelligence

Artificial intelligence and deep learning are also thoroughly dependent on the progress in computer science. Initially, many outstanding researchers in this domain believed that it would only be a question of decades before the intelligence of a human being could be matched by machines. Even Turing was confident that it would be possible for a machine to pass the Turing Test by the end of the century. Some scientists believe that we are not far from that point. Even though most problems in AI turned out to be algorithmically hard, since they are closely related to combinatorial problems. This led to an intensive research of heuristics and "soft" procedures – methods we humans use daily to solve complex problems. The combination of these methods and the computer's extraordinary speed in symbolic manipulation produces a powerful means to implement complex problems in AI.

Physics and engineering

Other scientific communities had already developed highly efficient procedures for solving sophisticated numerical problems before the first computer was built. This is especially true in physics and engineering. For example, Eugène Delaunay (1816-1872) made a heroic effort to calculate the moon's orbit. He dedicated 20 years to this pursuit, starting in 1847. During the first ten years he carried out the hand calculation by expressing the differential sys-

[1] Human calculators carrying out extended reckoning were called "computers" until the end of the 1940's. Many research laboratories utilized often poorly paid human calculators – most of them were women.

tem as a lengthy algebraic expression in power series, then during the second ten years he checked the calculations. His work made it possible to predict the moon's position at any given time with greater precision than ever before, but it still failed to match the accuracy of observational data from ancient Greece. A hundred year later, in 1970, André Deprit, Jacques Henrard and Arnold Rom revised Delaunay's calculation using a computer-algebra system. It took 20 hours of computer time to duplicate Delaunay's effort. Surprisingly, they found only 3 minor errors in his entire work.

Many numerical algorithms, such as the Runge-Kutta algorithm and the Fast Fourier Transform (FFT), were already known – the later even by Gauss – before the invention of the computer and they were much used by human "computers". But for many problems these efforts were hopeless, for the simple reason that the human computer was too slow to execute the simple but lengthy arithmetics.

Meteorology

An interesting illustration of this point is the origin of numerical meteorology. Prior to World War II, weather forecasting was more of an art, depending on subjective judgement, than a science. Although, in 1904, Vilhelm Bjerknes had already elaborated a system of 6 nonlinear partial differential equations, based on hydro- and thermodynamic laws, to describe the behavior of the atmosphere, he recognized that it would take at least three months to calculate three hours of weather. He hoped that methods would be found to speed up this calculation. The state of the art did not fundamentally change until 1949 when a team of meteorologists – encouraged by John von Neumann, who regarded their work as a crucial test of the usefulness of computers – fed the ENIAC with a model and got a 24-hour "forecast", after 36 hours of calculations, which turned out to be surprisingly good. Four years later, the Joint Numerical Weather Prediction Unit (JNWPU) was officially established; they bought the most powerful computer available at that time, the IBM 701, to calculate their weather predictions. Since then, many more complex models have been introduced. The computer has transformed meteorology into a mathematical science.

Economics

In still other scientific communities, until very recently, mathematical modeling was not even a topic or was used in a purely academic manner. Economics is a good example of this. Economists produced many nice models without practical implications. Sometimes, such models have even been developed just to give more credence to policy proposals. But mathematical models are not more credible simply because they are expressed in a mathematical way. On the other hand, important theoretical frameworks – such as game theory going back to the late twenties when John von Neumann published his first article on this topic – have been developed. Realistic n-person games of this theory cannot be solved analytically. They need to be simulated on computers. So the attitude towards these formal methods is also gradually changing in these other sciences as well. Today, no portfolio manager works without optimizing software. Branches such as evolutionary biology which have been more philosophical, are also increasingly penetrated by mathematical models and methods.

1.4 The consequences

We need more persons with modeling skills

We should not underestimate the significance of the development of computers for mathematical modeling. Their capacity to solve mathematical problems has already changed the way in which we deal with and teach applied mathematics. The relative importance of skills for arithmetic and symbolic manipulation will further decrease. It will be more important to *understand* the concept of a derivative than to calculate it. And we will need more persons qualified to *translate real-life problems into formal language*, and what activity is more rewarding and intellectually more challenging in applied mathematics than just *that* – namely modeling? While relatively fewer mathematicians are needed to solve a system of differential equations or to manipulate certain mathematical objects – these are tasks better done by computers – more and more persons are needed who are skilled and expert in modeling – *in capturing the essence, the pattern and the structure of a problem and to formulate it in a precise way*.

This development is by no means confined to science. Since World War II, a growing interest has been shown in formulating mathematical models representing physical processes. These kinds of models are also beginning to pervade our industrial and economical processes. In several key industries, such as chip production or flight traffic, optimizing software is an integral part of their daily activities. In many other industrial sectors, companies are beginning to formalize their operations: Planning the workforce, sport scheduling, production planning, route planning for transportation, just to mention a few. Mathematical models are beginning to be an essential part of our highly developed society.

Science ≡ modeling ≡ problem solving

An important condition for the widespread use of mathematical modeling tools is a change in the mathematical curriculum in school: A greater part of the manipulation of mathematical structures should be left to the machine, but more has to be learnt about how to recognize a mathematical structure when analyzing a particular problem. It should be an important goal in applied mathematics to foster creative attitudes towards solving problem and to encourage the students' acquisition and understanding of mathematical concepts rather than drumming purely mechanical calculation into their heads. Only in this way can the student be prepared for practical applications and modeling. **Science is modeling!** The ability to solve problems is modeling!

How to learn mathematical modeling?

But how can modeling be learnt? Problems, in practice, do not come neatly packaged and expressed in mathematical notation; they turn up in messy, confused ways, often expressed, if at all, in somebody else's terminology. Therefore, a modeler needs to learn a number of skills. She must have a good grasp of the system or the situation which she is trying to model; she has to choose the appropriate mathematical methods and tools to represent the problem formally; she must use software tools to formulate and solve the models; and finally, she should be able to communicate the solutions and the results to an audience, who is not necessarily skilled in mathematics.

Learning by doing

It is often said that modeling skills can only be acquired in a process of learning-by-doing; like learning to ride a bike can only be achieved by getting on the saddle. It is true that the case study approach is most helpful, and many university courses in (mathematical) modeling use this approach. But

it is also true – once some basic skills have been acquired – that theoretical knowledge about the mechanics of bicycles can deepen our understanding and enlarge our faculty to ride it. This is even more important in modeling industrial processes. It is not enough to exercise these skills, one should also acquire methodologies and the theoretical background to modeling. In applied mathematics, more time than is currently spent, should be given to the study of discovery, expression and formulation of the problem, initially in non-mathematical terms.
So, the novice needs first to be an observer and then, very quickly, a do-er. Modeling is not learnt only by watching others build models, but also by being actively and personally involved in the modeling process.

2 What is a Model?

In this section, a short introduction of the informal concept of model and related concepts is given. Then a precise definition of mathematical model is exposed and short examples explain the concept.

2.1 Modeling – an Informal Definition

The term model has a variety of meanings and is used for many different purposes. We use *modeling clay* to form small replica of physical objects; children – and sometimes also adults – play with a *model railway* or model aeroplane; architects build (scale) *model houses* or (real-size) model apartments in order to show them to new clients; some people work as *photo models* , others take someone for a model, many would like to have a *model friend*. Models can be much smaller than the original (an orrery, a mechanical model of the solar system) or much bigger (Rutherford's model of the atom). A diorama is a three-dimensional representation showing lifelike models of people, animals, or plants. A cutaway model shows its prototype as it would appear if part of the exterior had been sliced away in order to show the inner structure. A sectional model is made in such a way that it can be taken apart in layers or sections, each layer or section revealing new features (e.g. the human body). Working models have moving parts, like their prototypes. Such models are very useful for teaching human anatomy or mechanics.

modeling clay

photo model

model friend

model of atom

The ancient Egyptians put small ships into the graves of the deceased to enable them to cross the Nile. In 1679, Colbert ordered the superintendents of all royal naval yards to build an exact model of each ship. The purpose was to have a set of models that would serve as precise standards for any ships built in the future. (Today, the models are exposed in the Musée de la Marine in Paris.) Until recently, a new aeroplane was first planned on paper. The next step was to build a small model that was placed in a wind-tunnel to test its aerodynamics. Nowadays, aeroplanes are designed on computers, and sophisticated simulation models are used to test different aspects of them.

ship models

The various meanings of *model* in the previous examples all have a common feature: a model is an imitation, a pattern, a type, a template, or an idealized object which *represents* the real physical or virtual object of interest. In the ship example, both the original and the model are physical objects. The purpose is always to have a copy of some original object, because it is impossible, or too costly, to work with the original itself. Of course, the copy is not perfect in the sense that it reproduces all aspects of the object. Only some of the particularities are duplicated. So an orrery is useless for the study of life on Mars. Sectional models of the human body cannot be used to calculate the body's heat production. Colbert's ship models were built so accurately that they have supplied invaluable data for historical research, this was not their original purpose. Apparently, the use made of these ship models has changed over time.

mental models

Besides these physical models, there are also *mental ones*. These are intuitive models which exist only in our minds. They are usually fuzzy, imprecise, and often difficult to communicate.

drawings

Other models are in the form of *drawings or sketches*, abstracting away many details. Architects do not generally construct scaled-down models. Instead, they draw up exact plans and different two-dimensional projections of the future house. Geographers use topographical models and accurate maps to chart terrain.

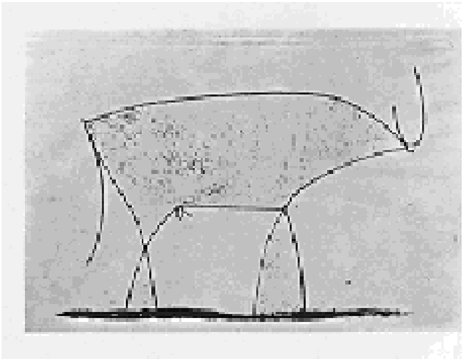

Figure 1: El Torro (P. Picasso, 1946)

Cave drawings are sketches of real animals; they inspired Picasso to draw his famous torro (see Figure 1). Picasso's idea was quite ingenious: how to draw a bull with the least number of lines? This exemplifies an essential characteristic of the notion of a model which is also valid for mathematical models, and for those in all other sciences: *simplicity*, *conciseness* and *aesthetics*. (More about these concepts see Chapter 2 of [10, 7].)
Certainly, what simplicity means depends on personal taste, standpoint, background, and mental habits. But the main idea behind simplicity and conciseness is clear enough: How to represent the object in such a way that we can "see" or derive its solution immediately? Our mind has a particular structure, evolved to recognize certain patterns. If a problem is represented in a way corresponding to such patterns, we can quite often immediately recognize its solution.

Intersection problem

A illustrative example is the *intersection problem* (from preface written by Ian Stewart in [2]). Suppose you have to solve the following problem (Figure 2): Connect the small square A with F, B with D, and C with E inside the rectangle by lines in such a way that they do not intersect.
At first glance, it seems that the problem is unsolvable, because if we connect A with F by a straight line, it partitions the rectangle into two parts, where B and C are in one part and D and E in the other. There is no way then, to connect B with D or C with E, without intersecting the line A-F. Well, we were not asked to connect A and F with a *straight* line. So let us connect A with F by a curved line passing between say, B and C. But, then again we

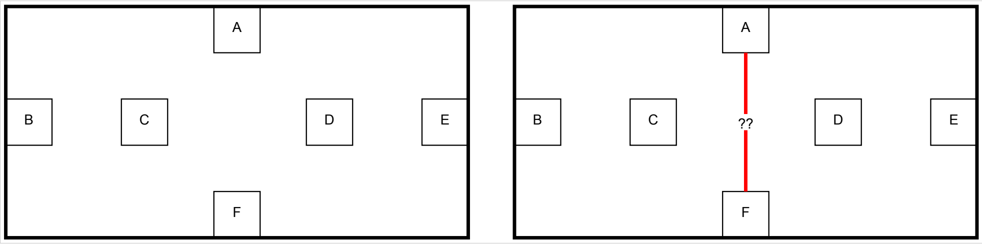

Figure 2: The Intersection Problem

have the same problem: The rectangle is partitioned into two parts, B being isolated. The same problem arises when the line from A to F is drawn between D and E. But the problem can be presented a little bit differently: Imagine that the rectangle surface is spanned by a thin rubber. So pick D and C (by two fingers) and rotate the rubber with the two fingers by 180 degrees around the center of the rectangle in a continuous way such that the places of C and D are interchanged (Figure 3).

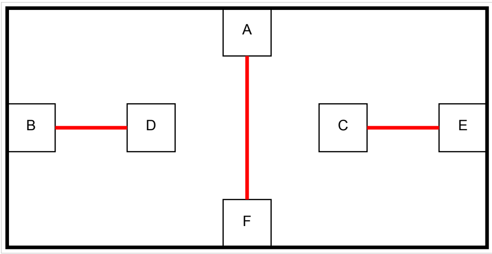

Figure 3: Topological Deformation

Now the problem is easily solvable. Connect the squares as prescribed. After this, return the rubber to the initial state again (Figure 4).
(A completely different approach to solve this problem can be found in my My Books – which is a good (advanced) exercise in mathematical modeling, see also the implemented model at intersec.)

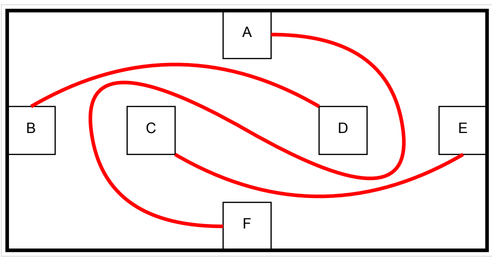

Figure 4: Solution to the Intersection Problem

Let's summarize:

- The model is the result of a *mapping* (or a transformation) of the real or virtual object to be considered.
- the model is an *abstraction*, i.e. only some aspects of the objects of interest are mapped, all others are ignored (abstrahere = leaving out).

- the aspects that enter the model are of *special interest* for an intelligent actor (the model-builder and the model-user): The model is used for a certain *purpose*.

- a model has some simple, concise (and aesthetic) *structure* that captures the focused aspects of the object.

Normally, the model is a simplification, but it must be rich enough to capture the problem at hand. Hence, it is something relative, there is no single absolute representation of the problem at hand. It is not only a model *of* something, but also *for* someone and *for* some purpose.

Physics

A good example is modern physics: Neglecting the Planck constant (that is: $\hbar = 0$), the gravitational constant ($G = 0$) and the speed of light ($c = \infty$), physics reduces to *classical mechanics*. Considering the gravitational constant ($G = 6.6741 \times 10^{11}[m^3/(kg \cdot s)]$) in addition, we must use *Newtonian mechanics*; Considering the speed of light ($c = 2.9979 \times 10^8[m/s]$ instead, we must apply *special relativity*, and so on (see [3]). It is not true, that the models of classical mechanics are learnt, because it is necessary for the "more advanced" theory as Quantum Field Theory. It is useful and applicable *for its own*, namely when the speed is small, when quantum effects are negligible, and when gravitation is not important – that is in our every day life. Classical mechanics has plenty of applications on its own. Each model in physics has its own eligibility and purpose, it will do to know its limitation.

Interpretations

Hence, we saw that a problem can be approached with different models. The contrary is also true: A (formal) model might have various *interpretations*. The intersection problem was built of a rectangle with 6 locations in it (A, B, C, D, E, and F) and connecting lines. So it is basically a *geometric interpretation*. Another interpretation is as following: Replace "rectangle" with "square electric board", "locations" with "endpoint of wires", and "lines" with "wires". Now we have the same model for a different problem: connect the endpoints of the wires on the 2-dimensional board in such a way that they do not cross. Another interpretation is: a oval closed room (with only one entry-point) has corresponding (curved) walls (from a point A to F, B to D and C to E, see the red lines in Figure 3) such that the whole room can be visited. Still another interpretation is: in an area we have six airports arranged in a similar way as the 6 locations in our original problem. Planify flight paths in such a way that the aeroplanes never cross each other. (We suppose that only the corresponding airports are linked, of course.)

An different question is whether the intersection problem is the right model for the flight-colliding problem. This could probably be solved in a more satisfactory manner by introducing different flying altitudes for the different connections. The wire-connection problem could probably be solved by insulating the wires and letting them intersect.

2.2 Mathematical models

This power of abstraction, this ability to leave out some or many details about a concrete problem in order to use the "distillate" for a multitude of similar problems, is one of the most striking characteristics of our mind. A concrete, "real physical" problem need not even be there: We can make some useful considerations without a specific application in mind, as we have seen with the intersection problem. Mathematics and logic are essential tools in this process of abstraction. They are powerful languages to express and represent real phenomena. We could even define mathematics as the *science of finding and representing patterns and structures* of concrete or abstract problems. They are these patterns that are expressed in a mathematical model.

So, what is a mathematical model? Informally, it is a list of *alternatives* fulfilling certain *criteria* from which we choose one, often the "best". Consider buying a house: we have a number of houses from which we choose one.

The buying-a-house problem

Formally, a mathematical model consists of symbols such as *operators* (arithmetic, Boolean, functions), *data* (the known entities), *variables* (the unknown entities), and *constraints* (the conditions). On a most basic level, we can define a mathematical model as a n-dimensional state space (or a simply as a set) fulfilling certain properties or conditions. Consider the problem of buying a house: the "state space" are all houses on the planet (or the houses in a certain region), the "variable" is the unknown house to choose, the "data" is our budget, etc. The "conditions or constraints" is our taste: "it must have at least 5 windows *and* 2 balconies", (*and* here is a Boolean operator).

In mathematics, we use symbols for these concepts:

Definition

$$Y = \{x \in X \mid R(x)\}$$

Y is called the mathematical model (which is a **set**), x is a vector of variables, X is a vector space of dimension n (often $\mathbb{R}^n$ or $\mathbb{Z}^n$, but not necessarily), R are the constraints, that is, a function which maps every element x in X into *true* or $false$: $R : X \to \{true, false\}$. By *solving a model* we means the find one or all $x^o \in X$ such that $R(x^o)$ is *true*. The problem is said *infeasible* if no such x^o exists, in other words if the set Y is empty, otherwise it is said the be *feasible*.
Consider our house-buying problem: X is the set of all houses: house1, house2, house3, ..., etc. X is one-dimensional in this case, such a list of all houses. x is one unknown house out of all houses. $R(x)$ is the required condition of each house x, it is true or false for every house: house1=true $[R(x) = true, x = house1]$ (yes, it has at least 5 windows and 2 balconies), $R(house2$=false ((no, it is not true that it has at least 5 windows and 2 balconies), $R(house3)$=false, etc. Y is the list of all houses that have a true-property: $Y = \{house1, \dots\}$, the list of *eligible* houses.
Let us give several examples (please go carefully through these examples to understand the concepts well).

Example 1:

A trivial model: $x = 1$

$$x \in \mathbb{Z}^+ \;,\; R(x) \stackrel{def}{=} (x = 1)$$

The state space consists of all positive numbers ($\mathbb{Z}^+$). x is a singleton variable which must be a positive integer, and its value is 1 by definition of R, that is, we choose exactly one number which value is *true*, namely 1. Hence $Y = \{1\}$, the model is feasible and has exactly one solution, namely $x = 1$. (It is as if we have only one house with at least 5 windows and 2 balconies, that the unique one we choose!)

Example 2:[2]

A infeasible model

$$x \in \mathbb{Z}^+ \ , \ R(x) \stackrel{def}{=} (x = 1 \wedge x = 2)$$

x is a singleton variable which must be a positive integer, and its value is 1 and at the same time 2 by definition of R. Hence $Y = \{\}$ (the set Y is empty), the model is infeasible, because there is no x which value is 1 *and* 2 at the same time. This is a contradiction. (It is as if no house fulfills our requirements.)

Example 3:

All points in a 2×2 square

$$x \in \mathbb{R}^2 \ , \ R(x) \stackrel{def}{=} (|x_1| \leq 1 \ \wedge \ |x_2| \leq 1)$$

The state space is 2-dimensional (every tuple of two real numbers is a possible choice). x is a 2-dimensional vector variable ($x = (x_1, x_2)$) with the components x_1 and x_2, they must be real numbers each, and its absolute values must be smaller or equal 1 by definition of R. Hence, the model is given by $Y = \{(x_1, x_2) \mid -1 \leq x_1, x_2 \leq 1\}$, the model is feasible, and has an infinity number of solutions: every point in a 2-dimensional Euclidean space included in the 2×2 unit square is a solution: so: $(-0.5, 0.3)$, $(0, 0)$, $(0.1, 0.2)$, $(0.4, 1)$, etc. are solutions. (Keeping our analogy with the choosing a house problem: We choose a house (x_1) together with a garage (x_2.)

Example 4:

The model has two solutions

$$x \in \mathbb{R}^2 \ , \ R(x) \stackrel{def}{=} (x_2 = x_1 + 2 \ \wedge \ x_2 = x_1^2 - 1)$$

The state space is again 2-dimensional. x is a 2-dimensional vector variable $x = (x_1, x_2)$ with the components x_1 and x_2 that must be real numbers, and they fulfill the two conditions defined by R. The solutions are:

$$Y = \left\{ \left(\frac{1 + \sqrt{13}}{2}, \frac{5 + \sqrt{13}}{2} \right), \left(\frac{1 - \sqrt{13}}{2}, \frac{5 - \sqrt{13}}{2} \right) \right\}$$
$$= \{(2.3, 4.3) \ , \ (-1.3, 0.7)\}$$

The model is feasible, and has exactly two solutions. To find the solutions, we need to solve a quadratic equation. Geometrically, the problem can be drawn in a 2-dimensional space (see Figure 5), the two solutions are where the line meets the parable.

Some mathematical models are easy to solve (the examples so far), others are very difficult. For example the last conjecture of Fermat is a difficult model:

Example 5:

Fermat's last conjecture

[2]Note that the symbol $\wedge$ is the Boolean and operator. This means that both $x = 1$ and $x = 2$ must be true to make the whole expression true.

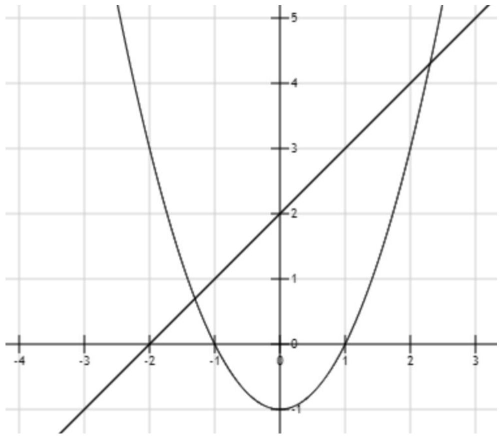

Figure 5: Geometric Solution

$$x = (a, b, c, n) \in \mathbb{Z}^4 \ , \ R(x) \overset{def}{=} (a^n + b^n = c^n \ \wedge \ n > 2)$$

This model is infeasible, that is, it has no solution. However, this 3-hundred year old conjecture has been proven by the mathematician A. Wiles only at 1997, after 9 years of intensive research.
Interestingly, a generalization by Euler, namely:

$$x = (a, b, c, d, n) \in \mathbb{Z}^5 \ , \ R(x) \overset{def}{=} (a^n + b^n + c^n = d^n)$$

has solutions. But it was not until 1960 to find one (others were found later on[3]):

$$95899^4 + 27519^4 + 27519^4 + 414560^4 = 422481^4$$

Even worse, it is easy to formulate models that cannot be solved because the computer takes far too much time to find a solution, for other models one can prove that they are unsolvable by any method.

Optimisation Models

A variant of a mathematical model is the *optimization problem*. In this case, we are not interested in any or all feasible solution, but only those solutions that maximize (or minimize) a function $f(x)$ (in analogy of the house choosing problem: we want the "best" house only):[4] *Definition*

[3] 1966: $27^5 + 84^5 + 110^5 + 135^5 = 155^5$ or 1988: $2682440^4 + 15365629^4 + 18796760^4 = 20615673^4$

[4] A optimization problem is commonly noted as

$$\begin{array}{ll} \text{min} & f(x) \\ \text{subject to} & R(x) \\ & x \in X \end{array}$$

I introduce a different notation to show clearly that a mathematical model is a *set* and this definition is more general:

$$Y = \{\min f(x) \mid x \in X \ \wedge \ R(x)\}$$

$$Y = \{\max f(x) \mid x \in X \ \wedge \ R(x)\}$$

Example 6: A trivial example is maximizing a parable in the whole real 1-dimensional space:

$$x \in \mathbb{R}\ ,\ f(x) = -(x-1)^2\ ,\ R(x) \stackrel{def}{=} (true)$$

The solution is $x = 1$, because the parable has a maximum at $x = 1$ where $f(x) = 0$ (for all other x the function value $f(x)$ is smaller than 0 (try it out!).

Example 7:

The can problem

Now let us construct a real but easy problem. Answer the following question "What is the height and the width of a cylindric aluminium can that contains a volume of 1 liter (dm^3), such that the quantity of material to build the can is as small as possible?"

Supposing that the material has identical thickness on its surface, we may substitute "quantity of material" by "surface of the cylinder". Given the height h and the diameter $2r$ of a cylinder, its volume is given as follows (which must be 1):

$$V = \pi h r^2 = 1$$

The surface O of the cylinder is then as follows (two circles and a cylinder coat):

$$O = 2\pi r^2 + 2\pi r h$$

We need to minimize the surface O (which we suppose to be proportional to the material usage, as already mentioned). Hence, the model is as follows:

$$x = (r, h) \in \mathbb{R}^{2+}\ ,\ \ \min f(x) \stackrel{def}{=} 2\pi r^2 + 2\pi r h\ ,\ \ R(x) \stackrel{def}{=} (\pi h r^2 = 1)$$

One can also simplify the model by eliminating the variable h. We then get the model:

$$O = 2\pi r^2 + \frac{2}{r}$$

where O must be minimal.

This is a minimization problem with one variable and no constraint. The solution is given in Figure 6. The minimal usage of material to construct a cylindric can that contains a volume of 1 liter has a radius of $5.42cm$ and a height of $10.9cm$, and the surface is $554cm^2$. Any other size has a larger surface, check it!

Another way to express the can problem – with two variables – is as follows:

$$\begin{array}{ll} \min & 2\pi r^2 + 2\pi r h \\ \text{subject to} & \pi h r^2 = 1 \\ & r, h \geq 0 \end{array}$$

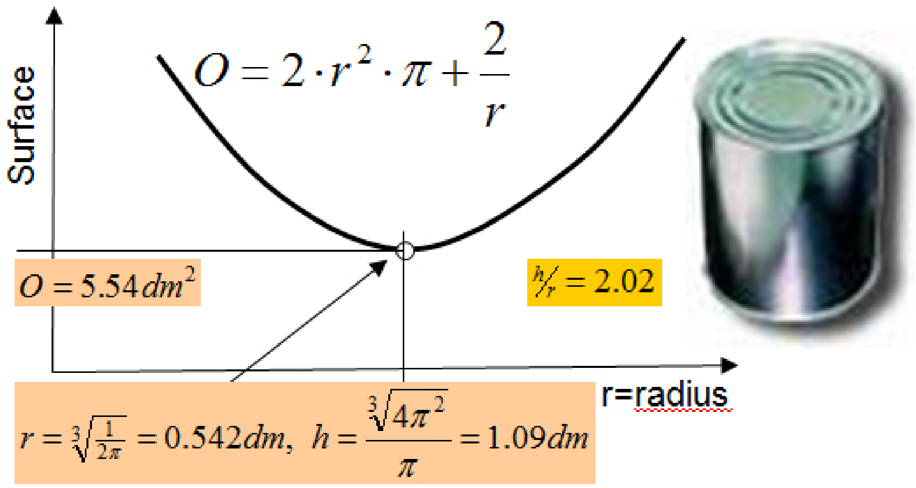

Figure 6: Solution of the can problem

Basically, every mathematical model is like our house choosing problem, only the size of the state space change and the requirements are expressed in a mathematical notation. At this point, I could introduce a really powerful concept to formulate *large and complex* mathematical models, that is, models with many variables and constraints. This concept is called *indexing*. Since it is a so important and powerful notation, I wrote a separate paper for that (see [9]). If you want to get a pro in mathematical modeling, these concepts are a must! *Indexing Paper*

A historical note: The concepts of the mathematical model are historically recent ones. According to Tarski, "the invention of variables constitutes a turning point in the history of mathematics." Greek mathematicians used them rarely and in an ad hoc way. Vieta (1540-1603) was the first who made use of variables on a systematic way. He introduced vowels to represent variables and consonants to represent known quantities (parameters). "Only at the end of the 19th century, however, when the notion of a quantifier had become firmly established, was the role of variables in scientific language and especially in the formulation of mathematical theorems fully recognized" [1] p.12.

2.3 Related concepts

For practical purposes, it is useful to define briefly some other related terms. The creator of a model is called the *modeler*. The modeler needs to have a profound knowledge of the object to be modeled, as well as an extensive understanding of mathematical structures in order to match them to the problem at hand. To create a model is and will always remain a complex and involved activity, even if computer based modeling tools can support the modeler, it cannot replace the creative work a modeler does. Of course, the modeler can be a team rather than a single person, as is often the case in big projects. In this case, the different activities such as data collection, setting up the model, solving the model, and verifying it, etc. can be distributed between the modelers. *The modeler*

Another concept is *modeling*. The term is difficult to define because it is used in so many different contexts. In art, it is used to express the process of working with plastic materials by hand to build up forms. In contrast to sculpturing, modeling allows corrections to be made on the forms. They can be manipulated, built and rebuilt until its creator feels happy with the result. In sculpturing, the objective cannot be formed, it must be cut out. Once finished, the form cannot be modified. Scientific activities are more like modeling than sculpturing, and therefore the common-sense meaning of modeling is used in various research communities. *Modeling*

In database technology, for instance, *data modeling* designates the design, validation, and implementation of databases by applying well-known methods and approaches such as the entity-relationship (ER) approach and the object-role modeling (ORM) method. From a practical point of view, data modeling could be considered as part of mathematical modeling. It is a set of tasks which consists of collecting, verifying, and ordering data.

Hence: *Modeling is an iterative process to conceptualize, to build, to validate, and to solve a model*. Often modeling is more important than the end-product, the model, to understand a problem. The creative process of getting though the modeling process gives invaluable insights in the structure of a particular problem.

(The concept of modeling process is so important that I have written a separate paper (see [8] (How to model?).)

Two further important notions in mathematical modeling are that of *model structure* and *model instance*. A model structure is a model where all data is captured by symbolic entities: the parameters.[5] In fact, it represents a whole class of models. A *model instance*, on the other hand, does not contain any parameters; these are replaced by concrete data. As an example, take a look at the following differential equation: *Model structure vs model instance*

$$\frac{dx}{dt} = \frac{rx(1-x)}{k}$$

The equation contains two parameters r and k (and a variable x). Hence, it defines an entire class of sigmoid functions and can, therefore, be used to model the growth of a population with limited resources. For a *concrete* growth model, however, the two parameters r and k must be replaced by numerical data. Additionally, an initial population x_0 must be given to make the equation numerically solvable.

Large models (models with many 1000 of variables and constraints) may also contain a large number of data that must first be collected, analyzed and verified. This may be a laborious task in itself, but that is another whole story.

Finally, a *modeling language* is a computer executable notation that represents a mathematical model augmented with some high level procedural statements to proceed the data of the model. There is an important difference between *Modeling language*

[5]The symbols with known values are called *parameters* and the symbols with unknown values are called *variables* – as already used above. Since these two terms clashes with well known concepts in programming languages, the convention is used to call the parameters in functions and procedures headings *formal parameters*. Using these parameters in a function or procedure call will be called *actual parameters*. The term variable used in programming languages, which is a name for a memory location, will be called *memory variable*.

programming and modeling languages. Whilst the former encodes an algorithm and can easily be translated to a machine code to be executed, the later *represents* a mathematical model. It does normally not have enough information in itself to generate a solution code. So, why should modeling languages be useful? First of all, such a computer executable notation encodes the mathematical model in a concise manner, and it can be read by a computer. The model structure can be analyzed and translated to other forms. Secondly, for many important model classes the notation can be processed and handed over to sophisticated algorithms which solve the problem without the intervention of a user. Modeling languages have many further benefits, for example, automatic model documentation, generating activity/constraint graph or similar graphical representations, etc. (There is another paper that explains my concept of a modeling language (see [6]).

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