Mixed Integer Programming to Schedule a Single-shift Workforce under Annualized Hours

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Abstract

Nowadays flexibility is a strategic concept for the firms. Indeed the workload has to follow, as closely as possible, the development of demand along the year. However, the firms cannot engage and dismiss employees according to the production requirements. Thus, the workforce scheduling becomes a delicate task. In this paper, different mixed integer programming models are developed in order to solve the workforce schedule problem for a single-shift. The annualized hour scenario is considered respecting a set of Swiss legal constrains. Furthermore, the minimal required workforce is guaranteed and it is assumed that each employee is able to perform each task within the team. All employees are full-time workers.

Keywords: Mixed integer programming, manpower planning, timetable, human resources.

1 Introduction

Last years companies were faced with the problem of fitting as well as possible the demand fluctuations and the market mutations with their workforce. This new reality had produced a wide number of new work models (annualized hours, job-sharing, job rotation, etc.) and intensified the use of existing ones (part-time working, temporary work, etc.), in order to bring the required flexibility to the firms (see [1], [2], [3]).

Different research studies have been focused on the workforce scheduling recently. The compressed work problem and the hierarchical workforce one were studied, for example, by Hung, Burns and Narasimhan (see references from [4] to [20]), whereas the annualized hours problem was analyzed by Hung, Azmat and Widmer (see [21], [22] and [23]). Mixed integer programming (MIP), as well as integer programming (IP), are well-known methods to solve scheduling problems. They were used to solve real problems in firms (see for example [24] and [25]) as well as the hierarchical workforce problem (see [26]).

This paper presents different MIP models in order to obtain the workforce schedule of a single-shift under annualized hours, taking into account a set of Swiss legal constrains. The aim is to generate a workforce schedule minimizing the overtime hours and balancing the employees's workload over a year. The minimal required workforce to assure the weekly demand is guaranteed.

In order to increase the flexibility of a firm, three strategies to spread the operating days over a week are developed. Each strategy gives rise to a MIP. Besides, each MIP includes two possible scenarios: in the first scenario, it is considered that each employee chooses his holiday weeks among a set of given weeks, whereas in the second scenario the MIP assigns the workers's holiday weeks among the same set of given weeks.

This paper is divided into six sections. Section 2 presents the problem description. Section 3 discusses a model for this problem. It describes the data, the variables and how the Swiss labor constrains were included into the model. Section 4 describes different MIPs based on the model of the problem. Section 5 presents the computational results of the MIPs and finally, section 6 discusses the use of the mixed integer programming in order to solve the annualized hours problem for a single-shift.

2 Problem Description

This paper focuses on Swiss companies active in the secondary sector and working with a single-shift under annualized hours. The annualized hour scenario implies that the number of work hours in a year can be irregularly spread over the weeks, but the annual amount of work hours is fixed. Employees earn the same wage each month. This scenario fits as well as possible the firm's workforce with the fluctuated demand over the weeks and decreases the number of overtime hours during the workload peaks.

Furthermore, in the context of this research, the annualized hours scenario also implies that the weekly workforce, the number of operating days in a week and the daily work length are computed according to the weekly demands of the firm and they keep its values along the week. These three values can change from one week to the other.

Employees are full-time workers and they are able to perform the different tasks within a team. The team starts its work in the morning and ends it during the afternoon. Work can be spread over 6 days of the week (from Monday to Saturday). Sunday is a day-off.

In order to increase the flexibility of the firms, three strategies for spreading the operating days over a week are discussed. Each strategy gives rise to a MIP:

- First strategy: the number of operating days identifies the last workday in a week. It is assumed that Monday is the first workday in a week, Tuesday the second one and so on until Saturday. For example, if the number of operating days is four, the firm will work from Monday to Thursday in this week.

- Second strategy: the management determines the operating days in a week. Thus, if the number of operating days in a week is four, the firm will work, for example, on Monday, Wednesday, Friday and Saturday.
- Third strategy: a MIP decides which are the operating days in a week. However, in order to fit this third strategy as closely as possible into the Swiss companies's reality, first of all the workdays will be assigned as operating days, whereas Saturday will only be an operating day if there must be six operating days in a week.

In order to bring this research as closely as possible to the companies's reality in Switzerland, a set of Swiss law constraints is considered:

L1: Each worker must receive at least one day-off per week. Sunday is normally considered as day-off;

L2: Each worker must receive at least 4 holiday weeks per year;

L3: Each worker has the right to receive at least two consecutive weeks of holidays (out of the four) a year;

L4: Companies fix the holiday periods, taking into account as far as possible the worker's individual wishes;

L5: The maximal work length per week is fixed to 45 hours for industrial workers, office employees, technical employees and staff of large companies. When the amount of worked hours is larger than 45 hours, the difference is considered as overtime hours. The normal work time is spread over 5 days a week;

L6: When annualized hours are considered, the workload can be spread irregularly over the year;

L7: The normal work duration per year is fixed to 2080 hours: 52 weeks with 40 hours per week (as each worker receives at least 4 holiday weeks, he works in fact a maximum of 1920 hours per year (48 weeks x 40 hours));

L8: A maximum of 225 hours per year and per worker can be considered as overtime hours; the cost of an overtime hour is 1.25 times the cost of a regular one;

L9: The work length can be spread between 06:00 AM and 8:00 PM. Each employee must receive at least one hour break, if his/her work length is over 9 hours a day and he/she must dispose at least of 11 consecutive hours-off a day.

Considering this set of law constrains, one can identify two types of overtime hours: the weekly overtime hours, defined by L5 and the annual overtime hours, defined by L8.

A holiday week is defined from Monday to Saturday for this research and two holiday scenarios are considered to spread the holiday weeks of the workforce over a year:

- First holiday scenario: each employee chooses his/her holiday weeks within a given set of weeks;

- Second holiday scenario: a MIP assigns to each worker at least four holiday weeks within the same set of weeks.

Each MIP arisen from an operating day strategy is computed with both holiday scenarios.

Finally, the aim of this research is:

- to determine the minimal workforce in order to fulfill the weekly demand;
- to minimize the number of weekly overtime hours;
- to minimize the number of annual overtime hours;
- to reach a balanced workload among the employees over a year and
- to generate the workforce schedule over a year.

3 Problem Modeling

In this section, it shows how to minimize the workforce in order to fulfill the weekly demand and how to minimize the number of weekly overtime hours thanks to the model of the problem.

This section is divided into three parts. The first one presents the data of the problem. The second part introduces the variables of the model and discusses the minimal yearly workforce in order to fulfill the demand. Part three explains how the minimal number of weekly overtime hours is computed (if weekly overtime hours are required).

Finally, even though the model of the problem presented in this section takes into account a set of Swiss law constraints, it is possible to adapt this model to other labor legislation (see [23]).

3.1 Data of the Problem

n	number of week in the planning period: 52 (see L7)
d_j	demand of week j, with j=1,..,52
h_{min}	minimal number of work hours in a week
h_{max}	maximal number of work hours in a week: 45 (see L5)
y_w	normal work length per worker over a year without holiday weeks: 1920 hours (see L7)
E_h	maximal number of overtime hours per worker over a year: 225 (see L8)
h	number of work hours per worker in a week ($h = h_{min}, .., m$ with $m \leq 72$, see L9)
M	matrix representing the scale of hours to work in a week.

The matrix M contains the link of the number of work hours per worker in a week (h), with the number of operating days and the shift length per operating day in a week. For example, if h is 36 in a week, these hours can be split, among other possibilities, in 4 workdays working 9 hours each or 5 workdays working 7.2 hours each. The management must introduce exactly one possibility to split each number of work hours in matrix M. Furthermore, the manager must first of all define the minimal number of work hours in a week (h_{min}) in order to fill this matrix M.

3.2 Variable of the Model

W	planning period workforce
w_j	workforce per operating day in week j ($w_j \leq W$)
K_j	number of operating days in week j
s_j	shift length per operating day in week j
ς	set of potential holiday weeks. ς guarantees at least four holiday weeks (and two consecutive one out of the four) for each employee

w_j must be computed for each week as follows:

$$w_j = min\{\lceil \frac{d_j}{h} \rceil \; with \; h \in [h_{min}, h_{max}] \; and \; such \; that \; (h{*}w_j - d_j) \; is \; minimized\}$$

If there are different values of h, which generate the same value of w_j, it is necessary to take the value of h minimizing the rest of the division. The variables K_j and s_j for each week are determined by the value of h according to the matrix M. Thus, the weekly work hours will always verify that $w_j * K_j * s_j \geq d_j$, with j=1,..,52.

Finally, the required minimal workforce satisfying the demand over the planning period (52 weeks) is calculated as follows:

$$W = max\{\lceil \sum_{j=1}^{52} \frac{w_j * K_j * s_j}{y_w} \rceil, max\{w_1, w_2, ..., w_{52}\}\}$$

This formula is similar to the one proposed by Hung [21].

The weekly workforce w_j follows as closely as possible the weekly demand of the firm, whereas the planning period workforce W guarantees the minimal required number of workers in order to fulfill each weekly workforce w_j.

3.3 Computing Weekly Overtime Hours

The workforce W satisfies the annual demand, because

$$W * y_w \geq \sum_{j=1}^{52} w_j * K_j * s_j \geq \sum_{j=1}^{52} d_j$$

However, this equation does not fulfill, the holiday constraints L2 and L3, in all cases. In order to satisfy these two legal constraints, the use of weekly overtime hours is required when necessary.

To fulfill legal constraint L2 (four holiday weeks per worker over a year), a first condition must be respected:

$$\sum_{j=1}^{52} (W - w_j) \geq 4 * W \qquad (1)$$

whereas, to guarantee two consecutive holiday weeks for each worker (L3), a second condition must be observed:

$$\varphi \geq W \qquad (2)$$

φ represents the number of pairs of consecutive weeks to allocate as holiday weeks.

If both conditions (1) and (2) are fulfilled, a feasible workforce planning is guaranteed. However, if at least one of these conditions is violated, it is necessary to choose the week ω, in which the corresponding demand can be satisfied in the most economical way with one worker less. This week ω is determined as follows:

$$\omega = argmin(\frac{w_j * K_j * s_j}{y_w}), \quad with \;\; 1 \leq \omega \leq 52$$

If different values of ω are available, it is necessary to choose the one, when possible, which increases φ, because this action implies automatically a decrease in the degree of violation of condition (1).

This operation (decreasing the number of workers of one unit for a given week) may lead to introduce weekly overtime hours and has to be repeated until conditions (1) and (2) are satisfied. In this way, a feasible workforce planning is guaranteed and the set of weeks ς is computed. Employees must choose their holiday weeks within this set ς.

4 MIP Formulation

This section presents six MIPs that solve the annualized hour problem. The first three resolve this problem taking into account each one different operating day strategy and the first holiday scenario, whereas all of other three MIPs also use one different operating day strategy but according to the second holiday scenario (see "Problem Description").

Some data presented in section three, as well as the variables computed in the same section ("Problem Modeling") are the data of each MIP.

Common data shared by the six MIPs:

W	planning period workforce
w_j	workforce per operating day in week j
K_j	number of operating days in week j
s_j	shift length per operating day in week j

y_w normal work length per worker over a year without holiday weeks ($y_w = 1920$; see L7)

E_h maximal number of overtime hours per worker over a year ($E_h = 225$; see L8)

Common variables shared by the six MIPs:

x_{ijk} if worker i works in week j during day k, this variable takes value 1, if not null

e_i represents the number of yearly overtime hours of worker i

Common constraints shared by the six MIPs:

$$(\sum_{j=1}^{52} s_j \sum_{k=1}^{R} x_{ijk}) - e_i \leq y_w \quad \text{with } i = 1, .., W \tag{1}$$

$$e_i \leq E_h \quad \text{with } i = 1, .., W \tag{2}$$

Workers' constraint (1) implies that each worker should not work more than y_w hours per year. R is equal to K_i in the first operating day strategy, whereas R is equal to 6 for the last two operating day strategies. The *yearly overtime hour constraint* (2) means that the number of yearly overtime hours per worker must be smaller than E_h.

Objective function of the six MIPs:

$$min\{[max(\sum_{j=1}^{52} s_j \sum_{k=1}^{R} x_{ijk}) - min(\sum_{j=1}^{52} s_j \sum_{k=1}^{R} x_{ijk})] + M * \sum_{i=1}^{W} e_i\}$$

This objective function minimizes, on the one hand the difference between the workloads of the worker with the maximal annual workload and his colleague with the minimal one and, on the other hand, the total number of yearly overtime hours of the workforce. The coefficient M, which represents a large positive number, allows to minimize the number of overtime hours before spreading the workload evenly among the employees over the year. R takes the value K_j in the first operating day strategy, whereas R takes 6 for the last two operating day strategies.

4.1 First Holiday Scenario

This subsection describes three mixed integer programs (MIP1, MIP2 and MIP3), which take into account the first holiday scenario: each employee chooses his/her holiday weeks within the set of weeks ς. Thus, a new data must be introduced in this context:

Specific data of MIP1, MIP2 and MIP3:

v_{ijk} worker i takes day k as off-day (holidays) in week j

Furthermore, a new constraint is introduced: if worker i takes holidays during day k in week j, he cannot work that day.

Specific constraint of MIP1, MIP2 and MIP3:

$$x_{ijk} = 0 \quad \text{if } v_{ijk} = 1 \tag{3}$$

4.1.1 First operating day policy and first holiday scenario (MIP1)

MIP1 takes into account that the weekly workforce works K_j consecutive days starting from Monday each week and that each worker chooses his holiday weeks within the set of weeks ς.

Data: w_j, K_j, s_j and v_{ijk}

Variables: x_{ijk} and e_i

MIP1

Objective function

$$min\{[max(\sum_{j=1}^{52} s_j \sum_{k=1}^{K_j} x_{ijk}) - min(\sum_{j=1}^{52} s_j \sum_{k=1}^{K_j} x_{ijk})] + M * \sum_{i=1}^{W} e_i\}$$

Constraints

$$(\sum_{j=1}^{52} s_j \sum_{k=1}^{K_j} x_{ijk}) - e_i \leq y_w \quad \text{with } i = 1,..,W \tag{1}$$

$$e_i \leq E_h \quad \text{with } i = 1,..,W \tag{2}$$

$$x_{ijk} = 0 \text{ if } v_{ijk} = 1 \quad \text{with } i = 1,..,W;\ j = 1,..,52 \text{ and } k = 1,..,K_j \tag{3}$$

$$\sum_{i=1}^{W} x_{ijk} = w_j \quad \text{with } j = 1,..,52 \text{ and } k = 1,..,K_j \tag{4.1}$$

$$x_{ijk} \in \{0,1\} \text{ and } e_i \in R^+$$

Workdays constraint (4.1) implies that exactly w_j workers are employed each day k during week j.

4.1.2 Second operating day policy and first holiday scenario (MIP2)

In MIP2, the management determines the operating days in a week, whereas each worker chooses his holiday weeks within the set of weeks ς.

Data: w_j, K_j, s_j, v_{ijk}

d_{jk} day k was defined by the management as operating day in week j

Variables: x_{ijk} and e_i

MIP2

Objective function

$$min\{[max(\sum_{j=1}^{52} s_j \sum_{k=1}^{6} x_{ijk}) - min(\sum_{j=1}^{52} s_j \sum_{k=1}^{6} x_{ijk})] + M * \sum_{i=1}^{W} e_i\}$$

Constraints

$$(\sum_{j=1}^{52} s_j \sum_{k=1}^{6} x_{ijk}) - e_i \leq y_w \quad \text{with } i = 1,..,W \qquad (1)$$

$$e_i \leq E_h \quad \text{with } i = 1,..,W \qquad (2)$$

$$x_{ijk} = 0 \text{ if } v_{ijk} = 1 \quad \text{with } i = 1,..,W;\ j = 1,..,52 \text{ and } k = 1,..,6 \qquad (3)$$

$$\sum_{i=1}^{W} x_{ijk} = d_{jk} * w_j \quad \text{with } j = 1,..,52 \text{ and } k = 1,..,6 \qquad (4.2)$$

$$x_{ijk} \in \{0,1\} \text{ and } e_i \in R^+$$

Workdays constraint (4.2) implies that exactly w_j workers are employed each workday k defined by the management in week j.

4.1.3 Third operating day policy and first holiday scenario (MIP3)

MIP3 decides on the operating days of each week taking into account that the workdays will be first of all assigned as operating days, whereas Saturday will only be an operating day if there are six operating days in a week. Each employee chooses his holiday weeks within the set of weeks ς.

Data: w_j, K_j, s_j and v_{ijk}

Variables: x_{ijk}, e_i

d_{jk} takes value 1 if day k is an operating day in week j; otherwise zero

MIP3

Objective function

$$min\{[max(\sum_{j=1}^{52} s_j \sum_{k=1}^{6} x_{ijk}) - min(\sum_{j=1}^{52} s_j \sum_{k=1}^{6} x_{ijk})] + M * \sum_{i=1}^{W} e_i\}$$

Constraints

$$(\sum_{j=1}^{52} s_j \sum_{k=1}^{6} x_{ijk}) - e_i \leq y_w \quad \text{with } i = 1,..,W \tag{1}$$

$$e_i \leq E_h \quad \text{with } i = 1,..,W \tag{2}$$

$$x_{ijk} = 0 \text{ if } v_{ijk} = 1 \quad \text{with } i = 1,..,W;\ j = 1,..,52 \text{ and } k = 1,..,6 \tag{3}$$

$$\sum_{k=1}^{6} d_{jk} = K_j \quad \text{with } j = 1,..,52 \tag{4.3}$$

$$\sum_{i=1}^{W} x_{ijk} = d_{jk} * w_j \quad \text{with } j = 1,..,52 \text{ and } k = 1,..,6 \tag{5.3}$$

$$d_{j6} = 0 \quad \text{if } K_j < 6 \tag{6.3}$$

$$x_{ijk}, d_{jk} \in \{0,1\} \text{ and } e_i \in R^+$$

Workdays constraint (4.3) implies that there are in week j exactly K_j workdays and *Workdays constraint* (5.3) guarantees w_j workers per workday of week j. Finally, constraint (6.3) expresses that Saturday of week j will be an off-day, if $K_j < 6$ in week j, in accordance with the third operating day policy.

4.2 Second Holiday Scenario

This subsection presents three mixed integer programs (MIP4, MIP5 and MIP6). Each MIP takes into account one different operating day policy and they allocate to each worker at least four holiday weeks within the set of weeks ς.

In order to assign the holiday weeks, two new variables are defined and four new constraints are introduced in MIP4, MIP5 and MIP6.

Specific variables of MIP4, MIP5 and MIP6:

- y_{ij} if worker i works at least one day during week j, this variable takes value 1, otherwise null
- z_{ij} if worker i works at least one day during two consecutive weeks (j and $j+1$), this variable takes value 1, otherwise zero

Holiday constraints of MIP4, MIP5 and MIP5:

$$\sum_{k=1}^{R} x_{ijk} \leq K_j * y_{ij} \quad \text{with } i = 1,..,W,\ j = 1,..,52 \tag{h1}$$

$$\sum_{j=1}^{52} y_{ij} \leq 48 \quad \text{with } i = 1,..,W \tag{h2}$$

$$y_{ij} + y_{i(j+1)} \leq 2 * z_{ij} \quad \text{with } i = 1,..,W,\ j = 1,..,51 \tag{h3}$$

$$\sum_{j=1}^{51} z_{ij} \leq 50 \quad \text{with } i = 1,..,W \tag{h4}$$

In constraint (h1) R takes the value K_j for MIP4, whereas R takes 6 for MIP5 and MIP6. *Holiday constraints* (h1) and (h2) guarantee that each employee will receive at least four holiday weeks over the year, whereas the other two *holiday constraints* (h3) and (h4) assign at least two consecutive weeks per worker. In equation (h4) 51 identifies the number of pair of weeks over the year.

4.2.1 First operating day policy and second holiday scenario (MIP4)

In MIP4, the weekly workforce works K_j consecutive days starting from Monday each week and MIP4 assigns the holiday weeks to the workforce within the set of weeks ς.

Data: w_j, K_j and s_j

Variables: x_{ijk}, e_i, y_{ij} and z_{ij}

MIP4

Objective function

$$min\{[max(\sum_{j=1}^{52} s_j \sum_{k=1}^{K_j} x_{ijk}) - min(\sum_{j=1}^{52} s_j \sum_{k=1}^{K_j} x_{ijk})] + M * \sum_{i=1}^{W} e_i\}$$

Constraints

$$(\sum_{j=1}^{52} s_j \sum_{k=1}^{K_j} x_{ijk}) - e_i \leq y_w \quad \text{with } i = 1,..,W \quad (1)$$

$$e_i \leq E_h \quad \text{with } i = 1,..,W \quad (2)$$

$$\sum_{i=1}^{W} x_{ijk} = w_j \quad \text{with } j = 1,..,52 \text{ and } k = 1,..,K_j \quad (4.1)$$

$$\sum_{k=1}^{K_j} x_{ijk} \leq K_j * y_{ij} \quad \text{with } i = 1,..,W,\ j = 1,..,52 \quad (h1)$$

$$\sum_{j=1}^{52} y_{ij} \leq 48 \quad \text{with } i = 1,..,W \quad (h2)$$

$$y_{ij} + y_{i(j+1)} \leq 2 * z_{ij} \quad \text{with } i = 1,..,W,\ j = 1,..,51 \quad (h3)$$

$$\sum_{j=1}^{51} z_{ij} \leq 50 \quad \text{with } i = 1,..,W \quad (h4)$$

$$x_{ijk}, y_{ij}, z_{ij} \in \{0,1\} \text{ and } e_i \in R^+$$

4.2.2 Second operating day policy and second holiday scenario (MIP5)

In MIP5, the management determines the operating days each week and MIP5 assigns the holiday weeks to the workforce within the set of weeks ς.

Data: w_j, K_j, s_j and d_{jk}

Variables: x_{ijk}, e_i, y_{ij} and z_{ij}

MIP5

Objective function

$$min\{[max(\sum_{j=1}^{52} s_j \sum_{k=1}^{6} x_{ijk}) - min(\sum_{j=1}^{52} s_j \sum_{k=1}^{6} x_{ijk})] + M * \sum_{i=1}^{W} e_i\}$$

Constraints

$$(\sum_{j=1}^{52} s_j \sum_{k=1}^{6} x_{ijk}) - e_i \leq y_w \qquad \text{with } i = 1,..,W \qquad (1)$$

$$e_i \leq E_h \qquad \text{with } i = 1,..,W \qquad (2)$$

$$\sum_{i=1}^{W} x_{ijk} = d_{jk} * w_j \qquad \text{with } j = 1,..,52 \text{ and } k = 1,..,6 \qquad (4.2)$$

$$\sum_{k=1}^{6} x_{ijk} \leq K_j * y_{ij} \qquad \text{with } i = 1,..,W,\ j = 1,..,52 \qquad (h1)$$

$$\sum_{j=1}^{52} y_{ij} \leq 48 \qquad \text{with } i = 1,..,W \qquad (h2)$$

$$y_{ij} + y_{i(j+1)} \leq 2 * z_{ij} \qquad \text{with } i = 1,..,W,\ j = 1,..,51 \qquad (h3)$$

$$\sum_{j=1}^{51} z_{ij} \leq 50 \qquad \text{with } i = 1,..,W \qquad (h4)$$

$$x_{ijk}, y_{ij}, z_{ij} \in \{0,1\} \text{ and } e_i \in R^+$$

4.2.3 Third operating day policy and second holiday scenario (MIP6)

MIP6 decides which are the operating days in a week taking into account that the workdays will be first of all assigned as operating days, whereas Saturday will only be an operating day if there are six operating days in a week. MIP6 assigns the holiday weeks to the workforce within the set of weeks ς.

Data: w_j, K_j and s_j

Variables: x_{ijk}, e_i, y_{ij} and z_{ij}

MIP6

Objective function

$$min\{[max(\sum_{j=1}^{52} s_j \sum_{k=1}^{6} x_{ijk}) - min(\sum_{j=1}^{52} s_j \sum_{k=1}^{6} x_{ijk})] + M * \sum_{i=1}^{W} e_i\}$$

Constraints

$$(\sum_{j=1}^{52} s_j \sum_{k=1}^{6} x_{ijk}) - e_i \leq y_w \quad \text{with } i = 1, .., W \quad (1)$$

$$e_i \leq E_h \quad \text{with } i = 1, .., W \quad (2)$$

$$\sum_{k=1}^{6} d_{jk} = K_j \quad \text{with } j = 1, .., 52 \quad (4.3)$$

$$\sum_{i=1}^{W} x_{ijk} = d_{jk} * w_j \quad \text{with } j = 1, .., 52 \text{ and } k = 1, .., 6 \quad (5.3)$$

$$d_{j6} = 0 \quad \text{if } K_j < 6 \quad (6.3)$$

$$\sum_{k=1}^{6} x_{ijk} \leq K_j * y_{ij} \quad \text{with } i = 1, .., W,\ j = 1, .., 52 \quad (h1)$$

$$\sum_{j=1}^{52} y_{ij} \leq 48 \quad \text{with } i = 1, .., W \quad (h2)$$

$$y_{ij} + y_{i(j+1)} \leq 2 * z_{ij} \quad \text{with } i = 1, .., W,\ j = 1, .., 51 \quad (h3)$$

$$\sum_{j=1}^{51} z_{ij} \leq 50 \quad \text{with } i = 1, .., W \quad (h4)$$

$x_{ijk}, y_{ij}, z_{ij} \in \{0, 1\}$ and $e_i \in R^+$

5 Computational Results

In order to test the six MIPs a set of 20 test-problems was generated. These 20 test-problems were assigned to each MIP. This section presents the computational results of the 120 MIPs.

Each problem was generated using the mathematical language LPL (see [28]) and solved by CPLEX 6.5 (see [29]), running on a computer Dell 800 MHz. The maximal running time to solve each MIP was 60 minutes.

This section presents two parts. The first one describes the generation of the test-problems and the second part exposes the results of the MIPs.

5.1 Test-problems Generation

Five values of W, the number of required employees over the year, were taken into account: 3, 5, 10, 15 and 20. There are two reasons in the choice of the upper bound: on the one hand, the mean size of a Swiss company active in the secondary sector is approximately 15 persons (see [27]) and, on the other hand, manufactures with more than 20 persons have mainly small teams of specific specialists. Furthermore, four workload intervals were chosen:]x.4, x.6],]x.6, x.8],]x.8, x.9],]x.9, W], where x = W-1.

Four important values were considered to generate the weekly demands over the year:

25 hours:	the minimal work duration per week (h_{min})
36.92 hours:	the normal work duration per year without the holiday weeks (y_w) spread over 52 weeks (1920 / 52; see L7)
40 hours:	the normal work duration per week (see L7)
45 hours:	the maximal work duration per week (see L5)

We have arbitrarily fixed h_{min} to 25. As the Swiss law fixes the maximal work duration per week to 45 hours, the value of $h_{min} = 25$ allow us to generate weekly demands in a wide context, producing examples with a fluctuated weekly demand over the year.

These four values define three intervals: A = [25, 36.92], B =]36.92, 40] and C =]40, 45]. To create the yearly demand, a mix of random weekly demands was generated in A, B and C.

The MIPs' data of each test-problem were generated taking into account the yearly demand and the model of the problem described in section 3. Table 1 shows the principal data of the 20 test-problems, where:

N:	problem number
Ex. W:	exact value of W : $\sum_{i=1}^{52} \frac{w_i * k_i * s_i}{y_w}$
max:	maximal weekly demand in the year [max $(w_j * K_j * s_j)$]
min:	minimal weekly demand in the year [min $(w_j * K_j * s_j)$]
ov.45:	weekly overtime hours

N	W	Interval	A	B	C	Ex. W	max	min	ov.45
1	3	]2.4, 2.6]	51	0	1	2.54	123.27	75.33	-
2		]2.6, 2.8]	43	4	5	2.67	131.31	75.33	-
3		]2.8, 2.9]	31	12	9	2.83	130.01	75.33	3
4		]2.9, 3.0]	18	25	9	2.98	131.31	75.33	12
5	5	]4.4, 4.6]	47	5	0	4.44	197.00	126.29	-
6		]4.6, 4.8]	37	12	3	4.66	219.63	126.29	-
7		]4.8, 4.9]	28	14	10	4.90	224.23	126.29	-
8		]4.9, 5.0]	25	18	9	4.95	224.21	127.16	-
9	10	]9.4, 9.6]	31	14	7	9.50	443.32	253.55	-
10		]9.6, 9.8]	28	13	11	9.64	449.23	257.61	-
11		]9.8, 9.9]	24	19	9	9.83	443.32	261.03	-
12		]9.9, 10.0]	20	22	10	9.99	446.16	256.23	-
13	15	]14.4, 14.6]	24	20	8	14.55	661.04	377.84	-
14		]14.6, 14.8]	25	15	12	14.72	668.98	377.84	-
15		]14.8, 14.9]	25	12	15	14.87	672.85	381.27	-
16		]14.9, 15.0]	25	10	17	14.94	672.85	381.27	-
17	20	]19.4, 19.6]	27	13	12	19.47	899.33	510.58	-
18		]19.6, 19.8]	25	13	14	19.76	899.33	510.58	-
19		]19.8, 19.9]	24	15	13	19.87	860.43	508.52	-
20		]19.9, 20.0]	23	15	14	19.93	890.17	510.58	-

Table 1: Principal data of the test-problems

	First holiday scenario	Second holiday scenario	Operating day strategy
Table 3	MIP1	MIP4	First
Table 4	MIP2	MIP5	Second
Table 5	MIP3	MIP6	Third

Table 2: Content of table 3, table 4 and table 5

5.2 Test-problems Results

Tables 3, 4 and 5 present a synthesis of the computational results of the test-problems. Each table exposes the results of the 20 test-problems using the same operating day policy under two holiday scenarios (see table 2).

Table 3, table 4 and table 5 show for each problem number (N):

max h wo :	maximal number of hours to work per worker in a year
min h wo :	minimal number of hours to work per worker in a year
$\triangle$:	[max h wo] - [min h wo]
ov y_w:	yearly overtime hours
$\triangle_\triangle$:	$\triangle_{MIPi}$ - $\triangle_{MIPj}$ with i=1, 2, 3 and j=4, 5, 6
S:	solver result

Column S presents one of the following values:

O: the solver has found the optimal solution of the problem;

M: the solver has found a feasible solution. There was not enough memory to find the optimal one;

T: the solver has found a feasible solution. The maximal runtime was reached.

Our first remark concerns the operating day policies. In fact, the balanced workload among the employees over the year is independent of the operating day strategies. In other words, the optimal solution of a problem will be the same for MIP1, MIP2 and MIP3 (see problem number 4 and 8 in table 3, table 4 and table 5) and will also be the same for MIP4, MIP5 and MIP6 (see problem number 4 in table 3, table 4 and table 5).

However, the two first operating day policies are important for the management: the first one imposes a block of operating days from Monday each week, while the second one allows the manager to decide on the operating days each week. The second policy is more flexible than the first one.

Finally, the third operating day policy, which allows a MIP to decide on the operating days each week, is interesting in a mathematical point of view.

Our second remark involves the holiday strategies. These have an important impact on the number of yearly overtime hours (see columns ov. y_w in table 3, table 4 and table 5 for problem number 12). In other words, the workers' individual wishes to choose their holiday weeks (see L4) can cost some overtime hours.

Our third remark has to do with the employees' workload over the year. In the context of the test-problem definition, we can observe that problems with a small workforce generally reach a better balance workload under the second holiday strategy (see column $\triangle_\triangle$ from problem number 1 until problem number 8 in each table), whereas the balanced workload of problems with a large workforce is normally better considering the first holiday strategy (see column $\triangle_\triangle$ from problem number 9 until problem number 20 in each table).

N	*MIP1*					*MIP4*					$\triangle_{\triangle}$
	max h wo	min h wo	$\triangle$	ov y_w	S	max h wo	min h wo	$\triangle$	ov. y_w	S	
1	1626.4	1626.2	0.2	0	T	1626.4	1626.2	0.2	0	T	0
2	1711.4	1711.2	0.2	0	T	1711.4	1711.2	0.2	0	T	0
3	1809.8	1808.5	1.3	0	T	1809.1	1808.9	0.2	0	T	1.1
4	1915.0	1903.5	11.5	0	O	1909.5	1908.5	1.0	0	O	10.5
5	1703.95	1703.6	0.35	0	M	1703.95	1703.65	0.3	0	T	0.05
6	1789.6	1788.6	1.0	0	M	1789.3	1789.05	0.25	0	T	0.75
7	1884.0	1880.2	3.8	0	M	1881.6	1880.4	1.2	0	T	2.6
8	1905.6	1901.2	4.4	0	O	1902.8	1902.4	0.4	0	T	4.0
9	1826.4	1823.4	3.0	0	T	1828.4	1819.8	8.6	0	T	-5.6
10	1852.6	1850.0	2.6	0	M	1854.0	1849.2	4.8	0	T	-2.2
11	1890.4	1885.2	5.2	0	M	1890.6	1885.6	5.0	0	T	0.2
12	1922.6	1915.4	7.2	6.2	T	1923.2	1915.2	8.0	4.2	T	-0.8
13	1865.4	1858.0	7.4	0	T	1868.8	1853.4	15.4	0	T	-8.0
14	1887.0	1879.6	7.4	0	T	1888.4	1879.0	9.4	0	T	-2.0
15	1907.0	1899.6	7.4	0	T	1905.4	1899.8	5.6	0	T	1.8
16	1915.6	1910.2	5.4	0	T	1915.6	1909.6	6.0	0	T	-0.6
17	1872.8	1865.6	7.2	0	T	1873.8	1864.6	9.2	0	T	-2.0
18	1900.6	1892.0	8.6	0	T	1902.0	1892.6	9.4	0	T	-0.8
19	1911.0	1902.8	8.2	0	T	1911.2	1901.4	9.8	0	T	-1.6
20	1918.6	1910.8	7.8	0	T	1920.0	1911.2	8.8	0	T	-1.0

Table 3: Synthesis of the results of MIP1 and MIP4

Problem number 4 is a clear example of the impact of the holiday strategies. All MIPs have found the optimal solution for this problem, however the workload difference ($\triangle$) is 11.5 hours in MIP1, MIP2 and MIP3, whereas this one falls down to 1 hour in MIP4, MIP5 and MIP6 (see columns $\triangle$ in table 3, table 4 and table 5 for problem number 4).

Our fourth comment is about the reduced number of yearly overtime hours. In fact, yearly overtime hours were only generated in problem number 12 (see column ov. y_w in table 3, table 4 and table 5). For this problem, table 1 shows that the exact value of the workforce (Ex. W)is 9.99 workers. The annualized hour model described in section 3 ("Problem Modeling") can generate yearly overtime hours, if the exact value of the workforce (Ex. W) is pretty close to the smallest integer number greater than itself.

N	*MIP2*					*MIP5*					$\triangle_\triangle$
	max h wo	min h wo	$\triangle$	ov y_w	S	max h wo	min h wo	$\triangle$	ov y_w	S	
1	1626.4	1626.2	0.2	0	T	1626.4	1626.2	0.2	0	T	0
2	1711.4	1711.2	0.2	0	T	1711.4	1711.2	0.2	0	T	0
3	1809.8	1808.5	1.3	0	T	1809.1	1808.9	0.2	0	T	1.1
4	1915.0	1903.5	11.5	0	O	1909.5	1908.5	1.0	0	O	10.5
5	1703.95	1703.6	0.35	0	M	1703.95	1703.65	0.3	0	T	0.05
6	1789.6	1788.6	1.0	0	M	1789.3	1789.05	0.25	0	T	0.75
7	1884.0	1880.2	3.8	0	M	1881.6	1880.4	1.2	0	T	2.6
8	1905.6	1901.2	4.4	0	O	1902.8	1902.4	0.4	0	T	4.0
9	1826.4	1823.4	3.0	0	T	1828.4	1819.8	8.6	0	T	-5.6
10	1852.6	1850.0	2.6	0	M	1854.0	1849.2	4.8	0	T	-2.2
11	1890.4	1885.2	5.2	0	M	1890.6	1885.6	5.0	0	T	0.2
12	1922.6	1915.4	7.2	6.2	T	1923.2	1915.2	8.0	4.2	T	-0.8
13	1865.4	1858.0	7.4	0	T	1868.8	1853.4	15.4	0	T	-8.0
14	1887.0	1879.6	7.4	0	T	1888.4	1879.0	9.4	0	T	-2.0
15	1906.2	1898.6	7.6	0	T	1905.4	1899.8	5.6	0	T	2.0
16	1915.6	1910.2	5.4	0	T	1915.6	1909.6	6.0	0	T	-0.6
17	1872.8	1865.6	7.2	0	T	1873.8	1864.6	9.2	0	T	-2.0
18	1900.6	1892.0	8.6	0	T	1902.0	1892.6	9.4	0	T	-0.8
19	1911.0	1902.8	8.2	0	T	1911.2	1901.4	9.8	0	T	-1.6
20	1918.6	1910.8	7.8	0	T	1920.0	1911.2	8.8	0	T	-1.0

Table 4: Synthesis of the results of MIP2 and MIP5

Finally, our last comment concerns the column S of tables 3, 4 and 5. In our case study, the solver found only the optimal solution of 8 problems over 120 ones. As well know, the number of variables and the constraint formulations are critical points for large size problems in order to find an optimal solution.

N	*MIP3* max h wo	min h wo	△	ov y_w	S	*MIP6* max h wo	min h wo	△	ov y_w	S	$\triangle_\triangle$
1	1627.0	1625.4	1.6	0	T	1626.4	1626.2	0.2	0	T	1.4
2	1712.8	1710.0	2.8	0	T	1711.4	1711.2	0.2	0	T	2.6
3	1809.8	1808.3	1.5	0	T	1809.1	1808.8	0.3	0	T	1.2
4	1915.0	1903.5	11.5	0	O	1909.5	1908.5	1.0	0	O	10.5
5	1704.1	1703.45	0.65	0	T	1704.3	1702.75	1.55	0	T	-0.9
6	1789.8	1788.8	1.0	0	T	1789.4	1788.6	0.8	0	T	0.2
7	1883.6	1880.2	3.4	0	M	1881.5	1881.0	0.5	0	T	2.9
8	1905.6	1901.2	4.4	0	O	1903.2	1902.2	1.0	0	T	3.4
9	1826.4	1823.4	3.0	0	T	1828.4	1819.8	8.6	0	T	-5.6
10	1852.6	1850.0	2.6	0	M	1854.0	1849.2	4.8	0	T	-2.2
11	1890.4	1885.2	5.2	0	M	1890.6	1885.6	5.0	0	T	0.2
12	1922.6	1915.4	7.2	6.2	T	1923.2	1915.2	8.0	4.2	T	-0.8
13	1865.4	1858.0	7.4	0	T	1868.8	1853.4	15.4	0	T	-8.0
14	1887.0	1879.6	7.4	0	T	1888.4	1879.0	9.4	0	T	-2.0
15	1907.0	1899.6	7.4	0	T	1905.4	1899.8	5.6	0	T	1.8
16	1915.6	1910.2	5.4	0	T	1915.6	1909.6	6.0	0	T	-0.6
17	1872.8	1865.6	7.2	0	T	1873.8	1864.6	9.2	0	T	-2.0
18	1900.6	1892.0	8.6	0	T	1902.0	1892.6	9.4	0	T	-0.8
19	1911.0	1902.8	8.2	0	T	1911.2	1901.4	9.8	0	T	-1.6
20	1918.6	1910.8	7.8	0	T	1920.0	1911.2	8.8	0	T	-1.0

Table 5: Synthesis of the results of MIP3 and MIP6

5.3 Conclusions

This paper discusses the annualized hours problem for a single-shift, taking into account a set of Swiss labor laws. Three operating day policies and two holiday strategies are discussed.

First of all, section 3 ("Problem Modeling") allows to compute the minimal required workforce to satisfy the demands and the number of weekly overtime hours. Then, the mixed integer programming technic was used to schedule the single-shift workforce, balancing the workload among the workers and minimizing the number of yearly overtime hours.

Although the mixed integer programming is a technic often used to solve scheduling problems, it is not always easy to find the optimal solution of a problem. However, feasible solutions proposed by the described MIPs are good enough to satisfy the requirement in balancing the workers' workload for Swiss companies active in the second sector. In the worst case, there are 15.4 hours between the worker with the maximal workload over the year and his colleague with the minimal one (see problem number 13 in table 3, table 4 and table 5). This represents less than 1 % of the normal work length per worker over a year without holiday weeks.

Finally, even if the mixed integer programming supplies high quality planning in the context of this case study, it will be wrong to believe that this technic can be tackled for all small and medium Swiss enterprises. Therefore, simple heuristics methods are an alternative approach to solve the annualized hour problem (see references [21], [22] and [23]).

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