

The 7-Digits Puzzle (Puzzle7)

Problem: Use each digit from 1 to 7 exactly once, and place them into the circles of Figure 1 in such a way that the sum along each of the five lines is the same.

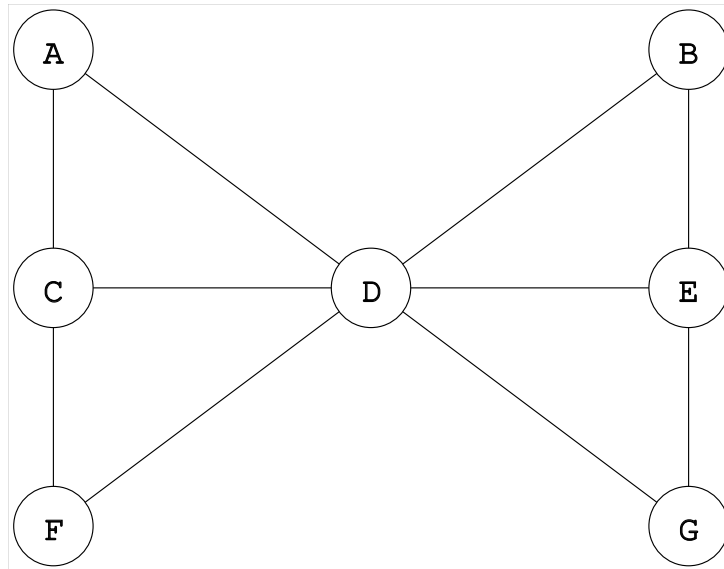


Figure 1: The 7-digit Puzzle

Modeling Steps: The key observations are: There are 7 circles and 7 digits. On each line we have 3 circles, hence 3 digits to place. Which digit should be at the center? Or: what numbers should be at the two ends of a line? Once answered these questions, it is easy to see the solution. Two observations lead to the number in the center: (1) The sum must be even, since otherwise at least four lines must have two even and one odd numbers, but there are only three numbers in $\{1, \dots, 7\}$ that are even. (2) the sum of all numbers is 28. The sum on the three lines through the center is $28 + 2D$ and it must be divisible by 3. Since the sum is even, $28 + 2D$ can only be 36. Hence the sum is 12 and $D = 4$.

Let's now formulate this problem as a mathematical model: Seven integer variables $x_i \in \{1, \dots, 7\}$ with $i, j \in I = \{A, \dots, G\}$ are to be determined and all must be different from each other. Furthermore, an additional variable z is introduced for the sum on each line.

Note that: Depending on the notation of the formulation, the sentence “all must be different from each other” could directly be interpreted as a predicate in the modeling formulation (as done here) or be translated as a set of inequalities as follows:

$$x_i \neq x_j \quad \text{forall } i, j \in I, i \neq j$$

In the LPL language the predicate is directly integrated as a “global constraint” with the keyword `alldiff` (another way in LPL is to use the function `Alldiff()`).

For each of the five straight lines a constraint is specified and the whole model can be formulated as follows :

$$\begin{aligned}
x_A + x_D + x_G &= z \\
x_A + x_C + x_F &= z \\
x_B + x_E + x_G &= z \\
x_B + x_D + x_F &= z \\
x_C + x_D + x_E &= z \\
x_i &\in \{1, \dots, 7\}, \text{ all different} \\
i &\in \{A, B, C, D, E, F, G\}
\end{aligned}$$

Listing 1: The Complete Model implemented in LPL [2]

```

model Puzzle7 "The 7-Digits Puzzle (puzzle)";
set i, j := ['A', 'B', 'C', 'D', 'E', 'F', 'G'];
alldiff variable x{i} [1..7];
variable z;
constraint
  L1: x['A'] + x['D'] + x['G'] = z;
  L2: x['A'] + x['C'] + x['F'] = z;
  L3: x['B'] + x['E'] + x['G'] = z;
  L4: x['B'] + x['D'] + x['F'] = z;
  L5: x['C'] + x['D'] + x['E'] = z;
solve;
Write('%s_ z=%d\n', {i} Format('%s=%d_', i, x), z));
--draw the solution
parameter X{i}:=[10 50 10 30 50 10 50];
           Y{i}:=[10 10 50 50 50 90 90];
set li{i,j}:=[('A', 'F') ('A', 'G') ('B', 'F') ('B', 'G') ('C', 'E')];
Draw.Scale(10, 5);
Draw.DefFont('Verdana', 20, 0, 2);
{li[i, j]} Draw.Line(X[i], Y[i], X[j], Y[j]);
{i} Draw.Circle(i & '=' & x, X, Y, 3, 1, 0);
--{i} Draw.Circle(i & '' , X, Y, 3, 1, 0);
end

```

Solution: The solution is given in Figure 2. The sum on each line is $z = 12$.

References

- [1] MatMod. Homepage for Learning Mathematical Modeling : <https://matmod.ch>.
- [2] Hürlimann T. Reference Manual for the LPL Modeling Language, most recent version. <https://matmod.ch/lpl/doc/manual.pdf>.

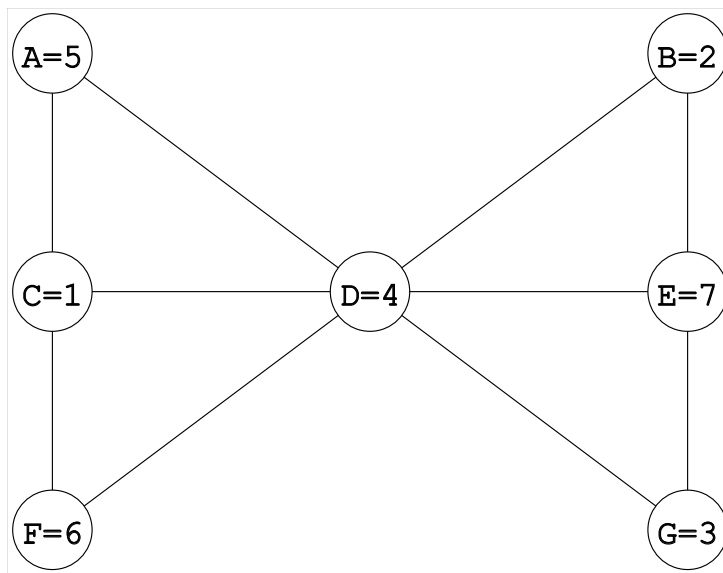


Figure 2: The Solution to the 7-digit Puzzle