Draw a Clock (DrawClock)

—- Run LPL Code , HTML Document –

Problem: This model draws a clock. It is the template for the following simple puzzle: "Divide the clock with a straight cut into two parts such that the sum of the numbers in both parts are equal ?". Can you *see* the solution at once?

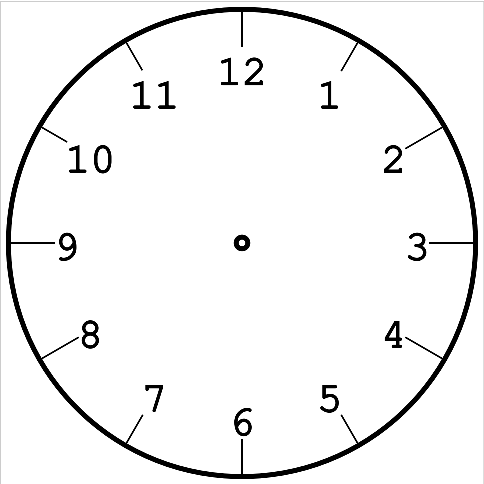

Figure 1: The Wall Clock

Modeling Steps

The key idea here is: Link the numbers 12 with 1, 11 with 2, 10 with 3, 9 with 4, 8 with 5, and 7 with 6 each by a straight line. The sum of the 6 pairs of matched numbers is always 13. Cut the clock into two parts between the 3-rd and 4-th line. On each side, the sum of the numbers is $3 \cdot 13$ (see Figure 2). Puzzle solved!

To prove that this is the unique solution, we must show that there is not another consecutive sequence of numbers on the clock face that sums to the half of all numbers. Let's choose two numbers $1 \leq x < y \leq 11$. The sum of all numbers on the face from $y + 1$ to x in clockwise direction is the sum from $y+1$ to 12 plus the sum from 1 to x (note that the sum of the numbers from 1 to n is given by $n(n + 1)/2$):

$$\left(\frac{12 \cdot 13}{2} - \frac{y(y+1)}{2}\right) + \frac{x(x+1)}{2}$$

This quantity must be half of all numbers from 1 to 12, that is it must be $\frac{12 \cdot 13}{4}$. Simplifying leads to:

$$(y - x) \cdot (y + x + 1) = 6 \cdot 13$$

As $y - x < y + x + 1$, we conclude that $y + x + 1 = 13$ and $y - x = 6$. Resolving for x and y gives: $y = 9$ and $x = 3$. Hence the unique solution is a consecutive sequence from 10 to 3 in clockwise direction.

Solution: This innocent simple problem can also be formulated as a mathematical model. It is a Knapsack problem with additional conditions. Given the set $i \in \{1, \ldots, 12\}$, let's introduces a binary variable x_i, where $x_i = 1$ if the number i is on one side of the cutting line, and $x_i = 0$ if it is on the other side.

1. Certainly, we must have that the sum of the numbers on one side must by half of the sum of all numbers, that is:

$$\sum_i i \cdot x_i = \left(\sum_i i\right)/2$$

This is in fact a Knapsack condition.

2. However, there is an additional requirement that the numbers on the two sides must be consecutive. We can express this by the following observation: take two consecutive numbers (modulo 12, that is, the next number of 12 is 1, etc.). There are 12 of such tuples, namely $(1,2), (2,3), \ldots, (12,1)$. Exactly for 2 of these tuples the condition holds that the variables x_i and the next consecutive variable, expressed as $x_{i \bmod 12+1}$, are different. This is the case when the cutting line passes. This fact can be expressed as a logical condition as follows:

$$\text{exactly}(2)_i\ (x_i \neq x_{i \bmod 12+1})$$

3. Solving the model gives $x_1 = 1$ for $i \in \{4, \ldots, 9\}$, as expected.

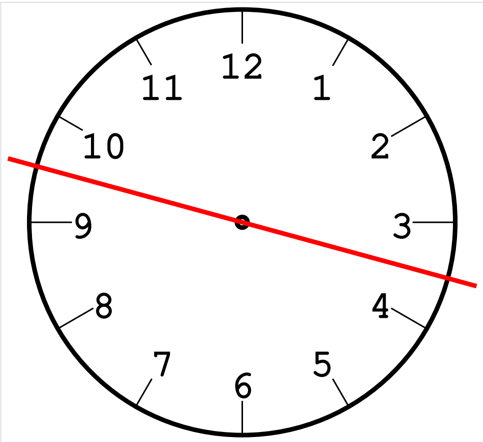

Figure 2: The Wall Clock, The Solution

Hence, the complete model is as follows:

$$\begin{aligned}
\sum_i i \cdot x_i &= \left(\sum_i i\right)/2 \\
\text{exactly}(2)_i\ (x_i &\neq x_{i \bmod 12+1}) \\
x_i \in \{0,1\} & \quad \text{forall } i \in \{1, \ldots, 12\}
\end{aligned}$$

The model implemented in LPL is straightforward :

```
model solveIt;
  set i:=1..12;
  binary variable x{i};
  constraint A: sum{i} i*x = (sum{i} i)/2;
  constraint B: exactly(2){i} (x[i] <> x[i%#i+1]);
  solve;
  Write('The␣numbers%3d␣are␣on␣one␣side\n',{i|x} i);
end;
```

Further Comments: Of course, one also can enumerate all possibilities. How many are their? 12 for cutting one number, 11 for cutting 2 (consequtive) numbers, 10 for cutting 3 (consequtive numbers, etc. That is,

$$\text{Number of all straight cuts } = 12 + 11 + 10 + \ldots + 1 = 12 \cdot 13/2 = 78$$

We could quickly eliminated most possibilities! However, the zest is to have a systematic procedure, and yes modeling is fun. No?

References

[1] MatMod. Homepage for Learning Mathematical Modeling : https://matmod.ch.